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CATHOLIC SCHOOLS

KEY IDEAS *for* Conceptual Development IN MATHEMATICS

2ND EDITION

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Foreword

This second edition of *Key Ideas for Conceptual Development in Mathematics* presents a comprehensive list of key ideas to support the teaching and learning of Mathematics in Catholic schools. It was developed by Learning Consultants, Mathematics, from Melbourne Archdiocese Catholic Schools (MACS) in consultation with Dr Vince Wright.

The resource aims to assist leaders and teachers of mathematics to plan for and implement key ideas into teaching and learning programs to enrich students' mathematical experiences and deepen their mathematical understanding.

Key Ideas for Conceptual Development in Mathematics is based on a culmination of research. I commend this guide to you and trust that it will be a valuable resource in your efforts to provide for the educational needs of all your students.

Jim Miles
Executive Director
Melbourne Archdiocese Catholic Schools

Preface

This resource is the second edition of *Key Ideas for Conceptual Development in Mathematics,* which was first published in 2013 by the Catholic Education Office in the Archdiocese of Melbourne.

This second iteration was developed by Melbourne Archdiocese Catholic Schools (MACS) Learning Consultants, Mathematics, in response to a need for leaders and teachers to understand the significance of the important ideas in mathematics.

Key ideas are central to the learning of mathematics. They link numerous mathematical understandings into a coherent whole, which supports learners in associating ideas and strategies, as opposed to seeing them as disconnected concepts, skills and facts.

Highly effective mathematics teachers understand the importance of key ideas in developing students' conceptual understandings in mathematics. They recognise that key ideas enable students to make deeper connections between the mathematical concepts and ideas that encompass and link them. When teachers plan for and implement key ideas into teaching and learning programs, the students have richer mathematical experiences and develop deeper mathematical understanding.

This second edition is based on a culmination of research and, like the first edition, includes full source information of all the studies cited in a references section. To increase the text's accessibility and enhance its user-friendliness, the team of mathematics learning consultants at MACS collaborated with Dr Vince Wright.

We would like to express our sincere gratitude to Vince for his tireless work in supporting us to make this second edition become a reality. He has provided invaluable advice and expertise in helping us shape this book to its final publication.

Learning Consultants, Mathematics
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Suggestions for using this resource

This resource aims to support leaders and teachers to think deeply about the ideas that underpin mathematical concepts. To enrich the mathematics being learned, it is recommended that this resource be used in conjunction with the Victorian Curriculum Mathematics, Victorian Curriculum and Assessment Authority Numeracy Learning Progressions and student data.

Mathematics leaders can use this resource when facilitating planning, leading professional learning meetings and promoting high-quality teaching of mathematics. Some areas they may choose to focus on include:

- building a shared understanding of conceptual development
- creating a culture of professional learning
- developing learning progressions of mathematics concepts
- deepening teachers' knowledge of mathematical and pedagogical content
- supporting teachers to identify and enact appropriate adjustments to facilitate student understanding in mathematics.

Teachers of mathematics can use this resource to plan for effective mathematics learning for their students. They may choose to use this resource to focus on:

- building a deep understanding of the mathematics curriculum
- identifying the important mathematics ideas embedded in the curriculum
- developing appropriate assessment tools
- selecting ways to determine students' mathematical skills and understanding.

OVERARCHING KEY IDEAS *for* ALL MATHEMATICAL CONCEPTS

The overarching key ideas for all mathematical concepts are: **estimation**, **benchmarks**, **visualisation**, **equality and equivalence**, **language** and **strategies**. These six key ideas have a broad application and are fundamental to enabling students to connect concepts across all areas of mathematics. Consequently, they need to be considered by educators when developing each unit of work. The overarching key ideas are outlined in more detail below.

Estimation

Estimation is an approximation or judgement of a value, quantity or measure.

An estimation is an educated approximation about a value that is as close to the exact value as is needed. All estimation is dependent on the estimator having benchmark numbers, facts or measures from which to work. Estimations may involve calculation, such as approximating the answer to 47×19 by rounding both numbers up to create 50×20. Estimation is also important in measurement.

Example: approximating the number of chairs that can be located in a room or knowing the area taken up by 10 chairs

Benchmarks

Benchmarks are trusted quantities or numbers used as reference points to estimate, calculate or compare.

Example: 100 and 50 might be used as benchmarks to estimate the result of subtracting 49 from 103.

Another example could be: is $\frac{5}{6}$ closer to 0, half or 1?

Visualisation

Visualisation is the making, storing, retrieval and manipulation of imagined objects and events.

These images can be true-to-life pictures of real-life objects or events, shapes, symbols, words and ideas associated with those objects or events.

Example: a young student might visualise two images of 9 and 6 in 10-frames, then imagine one object being moved to create 10 and 5.

Visualisation is important to a broad range of subject areas as it allows students to predict the result of actions in their head, without the need to necessarily carry out those actions.

Equality and equivalence

Equality and equivalence involve describing the relationship between two or more quantities as being 'the same as' in size, quantity, value, or in some other way.

Equality is important to arithmetic and algebra.

Example: the equation $13 = 7 + 6$ expresses the 'sameness' of quantity, as represented by the expressions on either side of the equals sign.

Equivalence is used in a similar way.

Example: $\frac{3}{4}$ and $\frac{9}{12}$ are equivalent fractions because they are different names for the same number.

Language

Language is specific vocabulary, graphics and symbols used to communicate mathematically with others.

It is used productively to create representations of ideas and receptively to interpret the ideas of others.

Language is an important tool for students to express mathematical concepts. Specialised mathematical language, such as 'factor', 'triangle' and 'average', embodies concepts that, in turn, can become ideas for students to use in their thinking.

Example: once students know the terms 'odd and even numbers', they can think about what happens when odd and even numbers are added or subtracted.

Symbols and diagrams, such as tables and graphs, provide means to represent, communicate and work with ideas in efficient and sophisticated ways.

Example: recording multiplication equations horizontally can highlight relationships in the facts:

$2 \times 12 = 24$

$4 \times 6 = 24$

$8 \times 3 = 24$

Strategies

Strategies are methods to solve mathematical problems.

They can be used: as general methods to solve problems (such as trial and improve or guess, check and refine); to solve a simpler, related problem; to make a table; or to look for a pattern. Strategies can also be specific to a type of problem.

Example: 28 + 39 can be solved by rounding both numbers, adding 30 and 40 to make 70, then taking away 3 to get 67.

KEY IDEAS *in* NUMBER AND ALGEBRA

Algebra: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in algebra. The specific key ideas in algebra are: **pattern**, **variable** and **function**.

Pattern

A *pattern* is a regularity or consistency in the arrangement of elements, such as in numbers, letters, shapes, objects or colours.

In early algebra, there are two main types of patterns:

- repeating patterns
- growing patterns.

A *repeating* pattern has a unit of repeat.

Example:

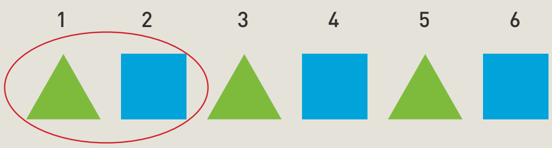


Figure 1

A *growing* pattern increases or decreases in a consistent manner. Number patterns are a type of growing pattern.

Example:

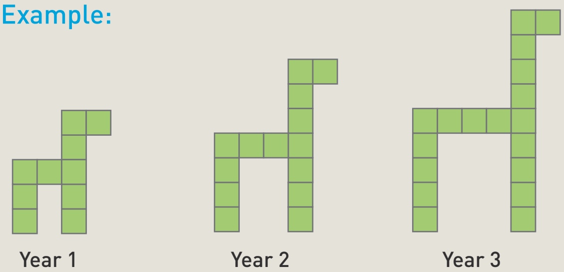


Figure 2

In mathematics, patterns are important because they can usually be expressed as generalisations. In the pattern shown in Figure 2, students might find a rule for progression of the pattern, such as: 'The llama grows by 4 tiles each year'. They might also find a direct rule for any year, such as: 'To find the number of tiles, multiply the year by 4 and add 6'.

Patterns can also be found in the behaviour of numbers when they are added, subtracted, multiplied or divided.

Example: two odd numbers always add up to an even number.

Variable

A *variable* is a letter or symbol used in algebraic expressions and equations. It is a letter that can have different values in the same problem. Letters are used most powerfully to represent relationships between variables (quantities that change).

Example: $7 + n = 15$, when $n = 8$.

Example: $\triangle + \lozenge = 16$, or $a \times b = 36$, where the values of $\triangle + \lozenge$, or the values of a and b, make the equation correct.

Example: $t = 4x + 6$, for the direct llama rule shown in Figure 2, where t = number of tiles.

Function

A *function* is a relationship that exists between variables. Each value of the input (independent variable) maps to only one value of the output (dependent variable). For example, the relationship between the number 7 and the number 5 is minus 2.

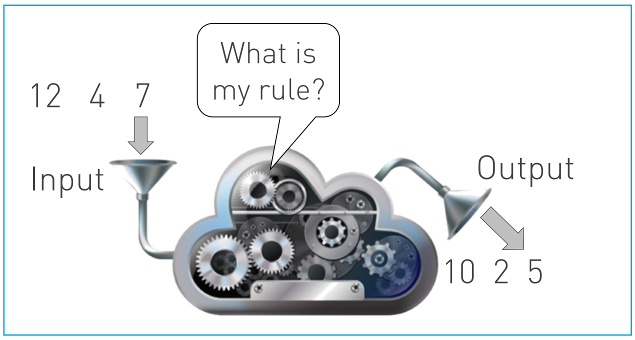


Figure 3

In the function Δ + 5 = ☐, each value of Δ results in one value of ☐.

Specific inputs and outputs can be written as ordered pairs.

Example: {(0,5), (1,6), (2,7), ...} are graphed on a number plane, as shown in Figure 4.

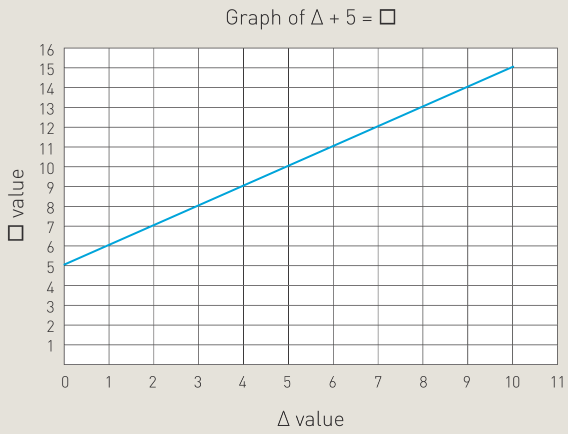


Figure 4

Algebra: important concept knowledge

Expression

An *expression* is a mathematical phrase with two or more terms (numbers or letters) connected by operations. Expressions do not have an equals sign.

Example A: 15 – 9
Example B: 8 × (2 + 4)

Equation

An *equation* is a mathematical statement with two or more expressions that are equal in value. An equation must have an equals sign.

Example A: 3 + 15 = 11 + 7
Example B: $x + 3 = 7$

Number properties

Number properties make an expression easier to work with and are applied when students perform calculations.

The most useful laws are the commutative, distributive and associative laws of addition and multiplication, as well as the inverse relationships between addition and subtraction, and between multiplication and division.

A student who solves 3 + 99 using 99 + 3 applies the commutative law of whole numbers under addition.

The following example uses the distributive and associative properties of whole numbers under addition.

Example:
9 + 6 = 9 + 1 + 5
= (9 + 1) + 5
= 10 + 5

Representations

Representations are artefacts used to code, create, investigate and communicate mathematical concepts and relationships. Important representations include:

- spoken and written language
- pictures or diagrams, including dynamic digital images
- manipulatives, sometimes called concrete materials
- symbols, including expressions and equations
- tables, including sequences and classifications by two variables
- graphs, including number planes.

Generalisation

A *generalisation* is a statement that holds true in all cases or for a specified set of cases. The process is: seeing features in examples, noticing a pattern, forming a conjecture about the pattern, justifying the conjecture using structure and generalising the rule.

Example: a student might notice that when three consecutive numbers are added, the sum is three times the middle addend, such as: 4 + 5 + 6 = 15, which is 3 × 5. They might form the conjecture that the pattern always works and try to structure the situation using algebraic or spatial representations. The final step is to prove that if Δ is the middle number, then any equation in the pattern can be expressed as (Δ -1) + Δ + (Δ + 1) = 3 × Δ.

Relational thinking

Relational thinking is the manipulation of both sides of an equality without needing closure. Equality is maintained if the same operation is applied to both sides of an equation.

Example: if 4 + 5 = 3 + 6, then 5 + 6 = 4 + 7.

In the above instance, one is added to each addend.

Relational thinking involves applying relationships within equations, across the equality, to find unknowns.

Example: ☐ + 12 = 15 + 10 is solved without calculating 15 + 10.

In the above example, since 12 is two more than 10, ☐ is two less than 15.

An equation is one way of showing equality.

Modelling

Modelling is making sense of mathematics in real-world situations by choosing and using appropriate mathematics.

A mathematical model is a representation of a situation that is created from data. The representation shows the relationships between variables. The purpose of a model is to make predictions about actions on the situation without needing to carry out the actions in reality.

Example: the equation 35 ÷ 7 = 5 models the number of weeks a packet of biscuits will last if consumed at one biscuit per day.

Order of operations

The *order* in which operations are carried out can affect the result. Therefore, the following convention is used:

- calculations within brackets are solved first
- in the absence of brackets, carry out operations in the following order:

  1. Powers
  2. Multiplication and division are solved working left to right
  3. Addition and subtraction are solved working left to right.

Example:

$$\begin{aligned}(5 + 4) \times 2 - 6 \div 3 &= 9 \times 2 - 6 \div 3 \\ &= 18 - 2 \\ &= 16\end{aligned}$$

Counting: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in counting. The specific key ideas in counting are: **quantity**, **abstraction principle**, **one-to-one principle**, **stable-order principle**, **cardinality principle**, **order-irrelevance principle**, **ordinal principle**, **conservation of number** and **subitising**.

Quantity

A *quantity* is an amount of something which is determined using a number and a unit.

The term 'manyness' applies to discreet quantities. When a set of discrete items is counted, the result assumes that the unit is one collection.

Example: five refers to 5 (number) teddies (unit) or 5 (number) fingers (unit).

The term 'muchness' applies to continuous quantities.

Example: 14 m describes a length that is measured to the nearest whole metre. The quantity 14.63 m describes a length that is measured to the nearest hundredth of a metre or centimetre. Both measures contain a number and a unit of measure.

Abstraction principle

In the *abstraction principle*, different sized or unrelated objects can be counted and treated the same numerically. Items that cannot be seen can also be counted – for example ideas, characters in a story, sounds, etc.

One-to-one principle

In the *one-to-one principle*, words in the forward or backward counting sequence are mapped onto the objects being counted; that is, there is one word to one object.

Stable-order principle

In the *stable-order principle*, there is a fixed order of words in the sequence when objects are counted.

Cardinality principle

In the *cardinality principle*, the last number indicates the total number of items; that is, it is a cumulative count.

Order-irrelevance principle

In the *order-irrelevance principle*, the order in which objects are counted does not change the quantity.

Example: counting the red objects first instead of the blue objects, does not change the number of overall objects.

The above five principles are from the work of Gelman and Gallistel (1978).

Ordinal principle

In the *ordinal principle*, numbers are used to indicate the position of an object in a numerical sequence or order.

Example: a red triangle is the third object in the pattern, or Leisha finished fifth in the race.

Conservation of number

In *conservation of number*, the number of objects in a collection does not change as the spatial arrangement of the collection changes.

Example: in Figure 5, both rows have the same number of apples.

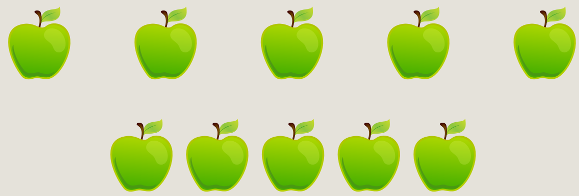


Figure 5

Subitising

Subitising is an instant recognition of a small quantity without counting.

Example: recognising a full hand of fingers as five

Counting: important concept knowledge

Number patterns

A *number pattern* is a regularity in a sequence of numbers. Number patterns are a type of growing pattern.

Examples:

skip counting – forwards and backwards, e.g. 5, 10, 15 ...

odd numbers

even numbers

square numbers, e.g. 1, 4, 9, 16 ...

Number word sequence

The *number word sequence*, forwards or backwards, is the fixed order of number names.

There is a difference between reciting and counting a number word sequence.

Reciting a sequence of number words is by rote, whereas counting is the allocation of each spoken number word with an item.

Students begin to learn the number word sequence and then use this sequence to count collections.

Whole number place value: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in place value. The specific key ideas in place value are: **quantity**, **number triad**, **digit position**, **base-10 system**, **place-value partitioning** and **comparison**.

Quantity

A *quantity* is an amount of something which is determined using a number and a unit.

Number triad

A *number triad* represents the relationship between words, symbols and materials or diagrams of a quantity.

Example: 35 might be represented as:

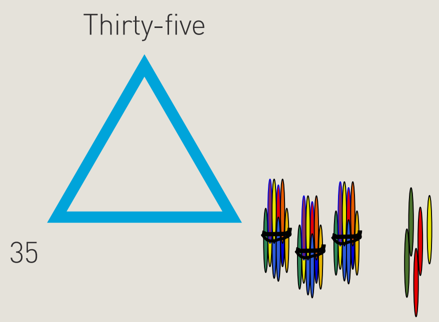


Figure 6

Digit position

The place of a digit in a number determines its value.

There are three ways to interpret a single digit within a whole number: *face value*, *place value* and *total value*.

Example: in 729, the seven has a face value of 7, a place value in the hundreds place, and a total value of 700.

Zero can be a placeholder and a digit representing a quantity.

Base-10 system

The *base-10 system* is a number system that is based on grouping and equally partitioning quantities by tens.

Each place has a value that is 10 times greater than the place to its right, and one-tenth of the value of the place to its left.

Example: 1000 is ten times more than 100 and is one-tenth of 10 000.

Place-value partitioning

Place-value partitioning is breaking a whole number into place-value units.

Example: 836 = 800 + 30 + 6 (compact and expanded form)

Comparison

The relative size of two quantities can be compared, either to each other or a benchmark.

Example: when comparing 22 and 13, a calculation is not required because 22 has two tens and 13 has one ten, therefore 22 is larger than 13.

Whole number place value: important concept knowledge

Rounding

Rounding is a context-driven approximation used to make numbers easier to calculate. It is useful for estimation.

The convention for rounding is that numbers are rounded up when the digits to the right of the nominated 'place' are equal to or greater than 5, and numbers are rounded down when these digits are less than 5.

Example: 66 can be rounded to the nearest ten (70) or to the nearest hundred (100).

Addition: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in addition. The specific key ideas in addition are: **combining**, **partitioning**, **part–part–whole** and the **properties of addition**. The properties of addition include: identity property, commutative property, associative property and inverse property.

Combining

Combining is the operation that represents the joining of two sets or quantities.

Partitioning

A quantity can be separated into parts while maintaining a sense of the whole.

Example: 10 is 5 and 5, 6 and 4, 5 and 4 and 1 etc.

Part–part–whole

A relationship exists between the parts and the whole. This relationship assists in finding the unknown quantity.

Example: 3 + ☐ = 7 and ☐ + 4 = 7 and 3 + 4 = ☐

Properties of addition

Addition has several properties, which are outlined below.

Identity property

Adding zero to a number will not affect the quantity. Zero is called the 'identity element' because it leaves the number unchanged.

Example: 17 + 0 = 17 or 0 +17 = 17

Commutative property

The order in which two numbers are added does not affect the sum.

Example: 6 + 3 gives the same sum as 3 + 6

Associative property

The order in which three or more addends are added does not affect the sum. Numbers can be arranged in different ways to make them easier to add.

Example: 2 + 9 + 8 could be solved by (2 + 8) + 9 or 2 + (8 + 9)

Inverse property

Addition and subtraction are related operations that undo each other, therefore subtraction can be used to solve an addition problem.

Example: 5 + 8 = 13 and 8 + 5 = 13, so 13 – 8 = 5 and 13 – 5 = 8

The inverse property is applied to form fact families.

Addition: important concept knowledge

Meaning of the numbers

Addition equations have at least two addends (numbers being added) and a sum (total).

Example: in 5 + 8 = 13, 5 and 8 are the addends and 13 is the sum. The + symbol represents the combining of the two quantities and the = symbol represents the equality (sameness) of 5 + 8 and 13.

Addition strategies

Addition strategies are methods to solve mathematical problems. With addition, the strategies may be mental, written, digital or a mix of the three.

Mental strategies are calculations worked in one's mind and may involve using one of the following methods:

- partitioning and recombining numbers, usually using place-value structure (split strategy)

Example: 48 + 23 = (40 + 20) + (8 + 3)

- jumping forward from a given number (jump strategy)

Example: 48 + 23 as 48 + 20 = 68, 68 + 3 = 71

- jumping strategies can be represented on an empty number line

Example:

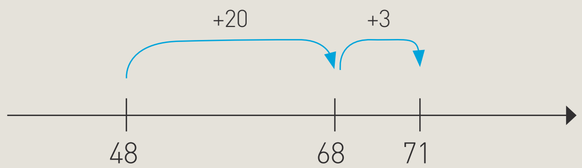


Figure 7

- rounding and adjustment strategies

Example: 39 + 48 as 40 + 50 – 1 – 2 (note that 39 is rounded to 40 and 48 is rounded to 50)

- transformation strategies, involving the shifting of a quantity from one addend to another

Example: 97 + 76 as 100 + 73 (three is shifted from 76 to 97)

- compensation strategies, involving the adjusting of one of the addends to make an equation easier to solve.

Example: 49 + 16 = 50 + 16 –1

Written strategies are often algorithms, meaning step-by-step methods to find an answer. The most common algorithm for addition applies place-value structure and should only be introduced once students have explored a range of other strategies and have developed a sound conceptual understanding of addition.

Example:

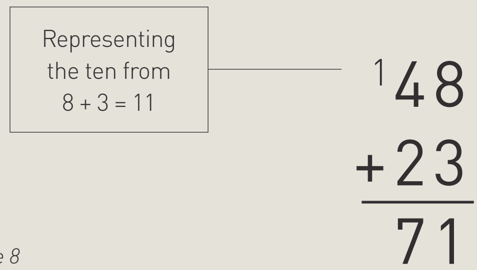


Figure 8

Part–part–whole

The unknown in result, change and start

Result unknown, *change unknown* and *start unknown* refer to different locations of the unknown in arithmetic problems.

Result-unknown problems have the answer as the result of the action.

Example: Kim has 7 Easter eggs. She gets 3 more eggs. How many Easter eggs are there altogether? (7 + 3 = ☐)

Change-unknown (missing addend) problems have an initial quantity and a result quantity, but ask for the change quantity.

Example: James has 7 Easter eggs. Nana gives him some more and now he has 10. How many Easter eggs did Nana give James? (7 + ☐ = 10)

Start-unknown (missing addend) problems ask for the beginning quantity.

Example: Jo has some Easter eggs. She gets 3 eggs from Nana and now she has 10. How many Easter eggs does Jo have before Nana gives her some? (☐ + 3 = 10)

Subtraction: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in subtraction. The specific key ideas in subtraction are: **separation**, **comparison**, **part–part–whole**, **partitioning** and **properties of subtraction**. The properties of subtraction include: identity property and inverse property.

Separation

Separation is subtracting or 'taking away' a quantity from a given collection.

Example: 7 take away 3 is 4.

Comparison

The relative size of two quantities can be compared and expressed as a difference.

Example: the difference between 7 and 3 is 4.

Part–part–whole

A relationship exists between the parts and the whole. This relationship assists in finding the unknown quantity.

Examples: 7 – □ = 4 and □ – 4 = 3 and 7 – 4 = □

Partitioning

A quantity can be separated into parts while maintaining a sense of the whole.

Example: 25 – 13 = (20 – 10) + (5– 3)
= 10 + 2
= 12

Properties of subtraction

Identity property

Subtracting zero from the minuend (initial quantity) has no effect on the difference.

Example: 5 – 0 = 5

Inverse property

Subtraction and addition are related operations that undo each other, therefore addition can be used to solve a subtraction problem.

Example: 12 – 5 = 7 so 5 + 7 = 12 and 12 – 7 = 5

The *inverse property* is applied to form fact families.

Subtraction: important concept knowledge

Meaning of the numbers

Subtraction equations that represent separation situations have a minuend (the whole collection) and a subtrahend (the part being removed) and a difference (the result).

Example: in 12 – 5 = 7, minuend – subtrahend = difference

Subtraction strategies

Subtraction strategies are methods to solve mathematical problems. The strategies may be mental, written, digital or a mix of the three.

Mental strategies are calculations worked in one's mind and may involve using one of the following methods:

- partitioning and recombining numbers, usually using place-value structure (split strategy)

Example: 78 – 45 = (70 – 40) + (8 – 5)

- jumping backwards from a given number (jump strategy)

Example: 83 – 46 as 83 – 40 = 43, 43 – 6 = 37 (note that jumping strategies are easy to represent on an empty number line, as Figure 9 shows)

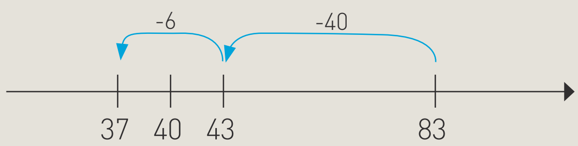


Figure 9

- rounding and adjustment strategies (compensation)

Example: 45 – 19 as 45 – 20 + 1 (note that 19 is rounded to 20 and 1 is added to adjust for subtracting too much)

- equal differences.

Example: 62 – 27 has the same difference as 65 – 30 or 60 – 25.

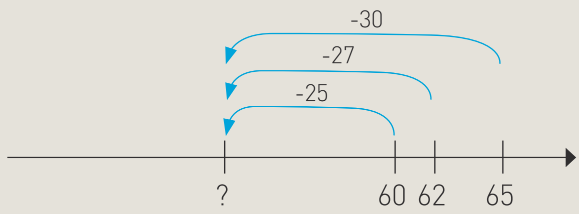
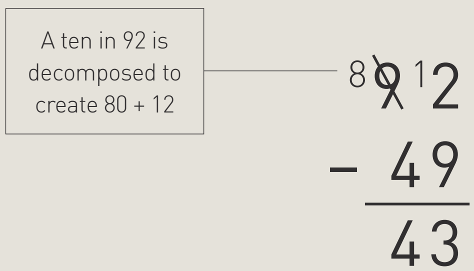


Figure 10

Written strategies are often algorithms, meaning they are step-by-step methods to find an answer. The two most common algorithms for subtraction are decomposition (of the minuend) and equal addition (to both the minuend and subtrahend). These two methods apply place-value structure and should only be introduced once students have explored a range of other strategies and have developed a sound conceptual understanding of subtraction.

Example:

Decomposition



Equal additions

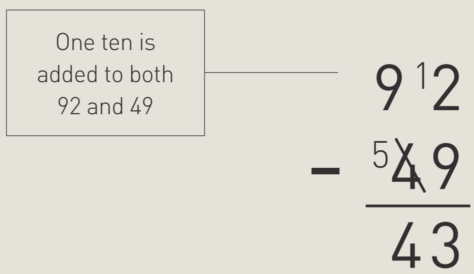


Figure 11

Part–part–whole

The unknown in result, change and start

Result unknown, *change unknown* and *start unknown* refer to different locations of the unknown in arithmetic problems.

Result-unknown problems have the answer as the result of the action.

Example: Kim has 7 Easter eggs. She eats 3 eggs. How many Easter eggs are left? (7 – 3 = ☐)

Change-unknown (missing subtrahend) problems have an initial quantity and a result quantity, but ask for the change quantity.

Example: James has 7 Easter eggs. He eats some of the eggs and now he has 4. How many Easter eggs does James eat? (7 – ☐ = 4)

Start-unknown (missing minuend) problems ask for the beginning quantity.

Example: Jo has some Easter eggs. She eats 3 of the eggs and now she has 4. How many Easter eggs does Jo have before she eats some? (☐ – 3 = 4)

Multiplication: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in multiplication. The specific key ideas in multiplication are: **equal groups**, **composite units** and **properties of multiplication**. The properties of multiplication include: commutative property, associative property, distributive property, null-factor property, identity property and inverse property.

Equal groups

The quantity in each group is the same.

Example:

Figure 12

Composite units

A *composite unit* is a collection of single items represented as one group.

Example: 6 ones is one group of 6.

Properties of multiplication

Multiplication has several properties, which are outlined below.

Commutative property

The order in which two numbers are multiplied does not affect the product.

Example: 3×6 results in the same product as 6×3.

Associative property

The order in which three or more factors are multiplied does not affect the product.

Example: $(3 \times 4) \times 5$ results in the same product as $3 \times (4 \times 5)$.

Distributive property

Factors can be partitioned, multiplied separately and the partial products are then added.

Example: $24 \times 6 = (20 + 4) \times 6 = (20 \times 6) + (4 \times 6)$ (note that 24 has been 'distributed' into $20 + 4$)

Null-factor property

Multiplying a number by zero will always give a product of zero.

Example: $5 \times 0 = 0$ represents the total of five sets with no objects in each set.

Identity property

Multiplying a number by one will not affect the quantity.

Example: $5 \times 1 = 5$ represents the total of five sets with one object in each set.

Inverse property

Multiplication and division are related operations that undo each other, therefore division can be used to solve a multiplication problem.

Example: $9 \times 7 = 63$, so $63 \div 7 = 9$, and $63 \div 9 = 7$

The *inverse property* is applied to form fact families.

Multiplication: important concept knowledge

Meaning of the numbers

Multiplication equations have a *multiplier* (how many groups or sets of equal size), a *multiplicand* (size of the equal sets) and a *product* (total number).

Example: in 9 × 7 = 63, 9 is the multiplier, 7 is the multiplicand and 63 is the product (the answer).

The × symbol means 'of' (as in equal 'sets' of) and the = symbol represents the equality (sameness) of 9 × 7 (nine 'sets' of seven objects) and 63 (objects).

Multiplication strategies

Multiplication strategies are methods to solve mathematical problems. The strategies may be mental, written, digital or a mix of the three.

Mental strategies are calculations worked in one's mind and may involve using one of the following methods:

- partitioning and recombining numbers, using the distributive property

Example: 6 × 24 as (6 × 20) + (6 × 4)

- rounding and adjustment strategies

Example: 5 × 38 as (5 × 40) – (5 × 2) (note that 38 is rounded to 40)

- proportional adjustment strategies.

Example: 6 × 24 = 12 × 12 (doubling and halving) or 3 × 27 = 9 × 9 (trebling and thirding)

Written strategies are often algorithms, meaning they are step-by-step methods to find an answer. The most common algorithm for multiplication applies place-value structure and the distributive property and should only be introduced once students have explored a range of strategies and have developed a sound conceptual understanding of multiplication.

Example:

Representing the 3 tens from 8 × 4 = 32 → ³54

$$\begin{array}{r} {}^{3}54 \\ \times \quad 8 \\ \hline 432 \end{array}$$

Figure 13

Multiplicative structures

Multiplication is applicable in a range of settings. These settings are sometimes referred to as 'problem types'. Table 1 provides some examples of problem types.

Table 1: Multiplicative structures

Multiplication problem type	Example
Equal groups	Each packet holds 6 biscuits. There are 5 packets. How many biscuits are there in total? (5 × 6 = 30)
Rate	Sally runs 3 laps every minute. How many laps will she run in 8 minutes? (8 × 3 = 24)
Times as many	Talia has four times as many marbles as Luke. Luke has 7 marbles. How many marbles does Talia have? (4 × 7 = 28)
Part–part–whole	For every 2 black cows in the herd, there are 3 brown cows. There are 100 cows overall. How many are brown? (2 + 3 = 5, 20 × 5 = 100, 20 × 3 = 60)
Cartesian product (combinations)	You have 4 different T-shirts and 3 different pairs of shorts. How many different outfits are possible? (3 × 4 = 12, or 4 × 3 = 12)
Rectangular array	Lettuces are planted in an array that is 9 lettuces long and 5 lettuces wide. How many lettuces are there? (9 × 5 = 45, or 5 × 9 = 45)

(Vergnaud 1988; Greer 1992)

Factors

A *factor* is a whole number that divides exactly into another number.

Example: 1, 2, 3 and 6 are factors of 6.

A prime number is a whole number greater than one with exactly two factors: itself and one.

A composite number is a whole number that has factors other than one and itself (ACARA 2019).

Multiples

A multiple is many groups of the same quantity.

Example: the multiples of 2 are 2, 4, 6, 8, 10, 12 and so on.

Division: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in division. The specific key ideas in division are: **equal groups**, **division with a remainder** and **properties of division**. The properties of division include: inverse property, identity property, division of a number by itself and division by zero.

Equal groups

The quantity in each group is the same.

Division with a remainder

A remainder occurs when a collection cannot be partitioned into equal groups.

Treatment of the remainder depends on the context. The remainder can be:

- discarded to give a smaller whole number

Example: how many full packets of 6 biscuits can be made from a tray of 25 biscuits? (4 full packets of biscuits)

- rounded up to the nearest whole number

Example: there are 29 students travelling to an AFL game. If only 3 students fit in each car, how many cars are needed? (10 cars)

- represented as a fractional part

Example: 15 worms are shared equally among 4 kookaburras. How many worms does each kookaburra get? ($23 \div 4 = 5\frac{3}{4}$)

- represented as a decimal.

Example: 22 metres of cloth makes exactly 5 dresses. How much cloth is needed for each dress? ($22 \div 5 = 4\frac{2}{5} = 4.4$ metres)

Properties of division

Inverse property

Division and multiplication are related operations that undo each other, therefore multiplication can be used to solve a division problem.

Example: $63 \div 7 = 9$ so $9 \times 7 = 63$

The *inverse property* is applied to form fact families.

Identity property

Dividing a number by one will not affect the quantity.

Example: $13 \div 1 = 13$ since there are 13 ones in 13.

Division of a number by itself

Dividing a number by itself will give a quotient (result) of one.

Example: $13 \div 13$ represents: how many groups of 13 can be made from a group of 13?

Division by zero

Dividing a number by zero is undefined.

Example: $13 \div 0$ represents: how many times can a group of 0 be made from a group of 13?

Division: important concept knowledge

Meaning of the numbers

The number which we divide is called the *dividend*. The number by which we divide is called the *divisor* and the result is called the *quotient*.

Types of division

Division takes two forms: partition division and quotition division.

Partition division

Partition division (or equal sharing division) is used when the total number to be divided is known (the dividend) and the number of parts is known. The number in each part is not known.

Example: I have 12 lollies and I share them into 4 bags. How many lollies are in each bag? (Think about: 'shared between'.)

Quotition division

Quotition division (also known as measurement division or repeated subtraction) is when the total number to be divided is known and the number in each part is known. The total number of parts is not known.

Example: I have 12 lollies and I put them into bags of 4. How many bags will there be? (Think about: 'how many groups of?')

Division in fraction form

The result of equal sharing or measuring can be represented as both a number and an operation.

Example: $\frac{3}{4}$ can represent both the operation 3 ÷ 4 and the quotient (answer).

Divisional structures

Division is applicable in a range of settings. These settings are sometimes referred to as 'problem types'. Table 2 provides some examples of problem types.

Table 2: Divisional structures

Division problem type	Partitive (sharing) example	Quotative (measuring) example
Equal groups	There are 30 biscuits and 6 packets. The biscuits are equally shared into the packets. How many biscuits are in each packet? (30 ÷ 6 = 5)	There are 30 biscuits. 5 biscuits are put in each packet. How many packets are there? (30 ÷ 5 = 6)
Rate	Sally runs 24 laps in 8 minutes. How many laps does she run each minute? (24 ÷ 8 = 3)	Sally runs 24 laps at 3 laps per minute. How many minutes does she run altogether? (24 ÷ 3 = 8)
Times as many	Talia has 28 marbles. That is four times as many marbles as Luke. How many marbles does Luke have? (28 ÷ 4 = 7)	Talia has 28 marbles and Luke has 7 marbles. How many times more marbles does Talia have than Luke? (28 ÷ 7 = 4)
Part–part–whole	There are 100 cows in the herd. For every 2 black cows there are 3 brown cows. How many brown cows are there? (2 + 3 = 5, 100 ÷ 5 = 20, 3 × 20 = 60)	In a herd of 100 cows there are 40 black cows and the rest are brown. What is the ratio of black to brown cows? (40 + 60 = 100, 40 ÷ 20 = 2, 60 ÷ 20 = 3 so 40:60 = 2:3)
Cartesian (product combinations)	You can make 12 different paired outfits with your T-shirts and shorts. You have 3 pairs of shorts. How many T-shirts do you have? (12 ÷ 3 = 4)	
Rectangular array	There are 45 lettuces planted in rows of 9 plants each. How many rows of lettuces are there? (45 ÷ 9 = 5)	

(Vergnaud 1988; Greer 1992)

Division strategies

Division strategies are methods to solve mathematical problems. The strategies may be mental, written, digital or a mix of the three. The strategies rely on modifying the properties of numbers under multiplication to allow for division as the inverse of multiplication.

Mental strategies are calculations worked in one's mind and may involve using one of the following methods:

- partitioning and recombining numbers using the distributive property, usually with place value

Example:

144 ÷ 6 (Partition 144)

120 ÷ 6 and **24 ÷ 6**

20 and **4**

24 (Recombine)

Therefore, 144 ÷ 6 = 24

Figure 14

- rounding and adjustment strategies

Example: 232 ÷ 8 = (240 ÷ 8) – (8 ÷ 8) (note that 232 is rounded to 240 because 240 is easily divided by 8)

232 ÷ 8 (Round 232 to a number that is divisible by 8, i.e. 240)

240 ÷ 8 and **8 ÷ 8** (Divide the adjustment by 8)

30 subtract **1**

29 (Recombine)

Therefore, 232 ÷ 8 = 29

Figure 15

- proportional adjustment strategies, using factors of the divisor

Example: 340 ÷ 5 = ?

340 ÷ 10 = 34 (Use the factors of 10 and proportionality)

÷ 2 **× 2**

340 ÷ 5 = 68 (Divisor is half, so the quotient is double)

Therefore, 340 ÷ 5 = 68

Figure 16

- equal adjustment to dividend and divisor to simplify an equation.

Example: 276 ÷ 12 = ?

276 ÷ 12 (Divide dividend and divisor by 3)

÷ 3

92 ÷ 4 (Divide dividend and divisor by 2)

÷ 2

46 ÷ 2 = 23

Therefore, 276 ÷ 12 = 23

Figure 17

Written strategies are sometimes algorithms, meaning step-by-step methods to get an answer. The most common algorithm for division applies place-value structure and the distributive property and should only be introduced once students have explored a range of strategies and have developed a sound conceptual understanding of division. Written strategies include:

Example: 64 divided by 4

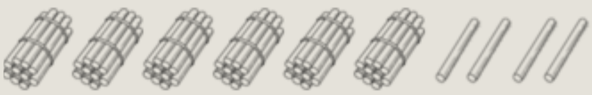


Figure 18

Share 6 tens among 4. That's 1 ten each.

4 tens have been shared so 2 tens are left to share. There are now 2 tens and 4 ones to share. As there are not enough tens to share, trade for ones.

There are now 24 ones to share.

Share 24 ones among 4. (Think: 4 × ? = 24)
Four sixes are 24, meaning 6 ones each.
So, 64 divided by 4 is 1 ten and 6 ones each which is 16.

4)	64	
	-40	10
	24	
	-24	6
	0	16

Figure 19

How many groups of 4 can be made from 60?
10 groups of 4 can be made, which is 40.
That leaves 24.
How many groups of 4 can be made from 24?
6 groups of 4 can be made, which is exactly 24 and no remainder.
64 divided by 4 is 10 groups of 4 plus 6 groups of 4 which is 16 groups of 4.

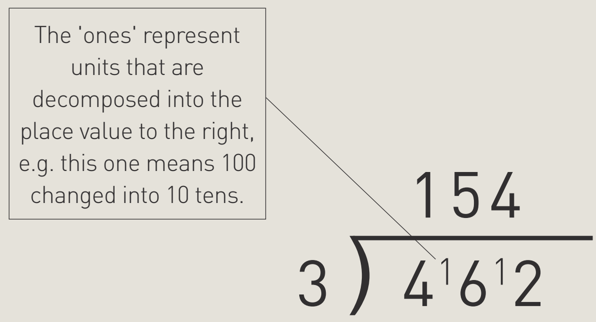


Figure 20

4 hundreds shared among 3 is 1 hundred each and 1 hundred remaining.
Rename the 1 hundred and 6 tens as 16 tens.
16 tens shared among 3 is 5 tens each and 1 ten remaining.
Rename the 1 ten and 2 ones 12 ones.
12 ones shared among 3 is 4 ones each and no remainder.

Misconception

A common misconception is that 'division makes smaller'. Since learning about division usually starts with separating a larger group into equal smaller groups, students often think the quotient is always smaller than the dividend.

Example: in 36 ÷ 4 = 9, 9 is smaller than 36.

The overgeneralisation that 'division makes smaller' causes problems with rational numbers.

Example: $6 \div \frac{1}{4} = 24$ since there are 24 quarters in 6.

Fractions: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in fractions. The specific key ideas in fractions are: **quantity**, **number triad**, **partitioning**, **equivalence** and **benchmarks**.

Quantity

A *quantity* is an amount of something which is determined using a number and a unit.

Number triad

A *number triad* represents the relationship between words, symbols and materials or diagrams of a quantity.

Example: five-sixths may be represented as:

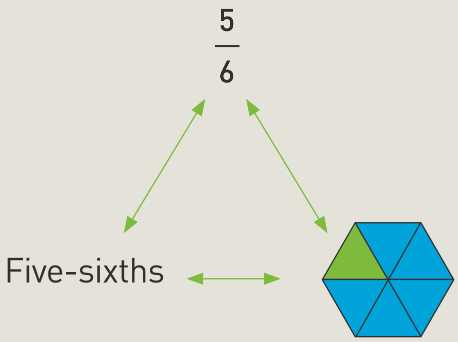


Figure 21

Partitioning

A quantity can be separated into parts while maintaining a sense of the whole.

Example: $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$

Equivalence

Fractions are equivalent if they represent the same quantity; for example $\frac{3}{4}$ is the same as $\frac{6}{8}$, $\frac{9}{12}$ ...

A fraction can also be represented as:

- a decimal – $\frac{3}{4}$ is the same as 0.75
- a percentage – $\frac{3}{4}$ is the same as 75%
- a ratio – $\frac{1}{2}$ is the same as 1:2.

Benchmarks

Benchmarks are trusted quantities or numbers used as reference points to estimate, calculate or compare.

Example: $\frac{2}{3}$ is closer to 1 than 0.

Fractions: important concept knowledge

Meaning of the symbols

In any fraction, the top number is called the *numerator* and the bottom number is called the *denominator*. The horizontal line that separates the numerator from the denominator is called the *vinculum*. The denominator indicates the size of the parts. The numerator indicates the number of parts of that size.

Example: in $\frac{4}{5}$ the numerator is 4 and the denominator is 5. The numerator is a count, so in $\frac{4}{5}$ there are 4 units. The denominator, 5, indicates the size of the parts; they are the units created when one (whole) is partitioned into five equal parts or fifths.

Fraction types

Proper fraction

The numerator is less than the denominator, e.g. $\frac{4}{5}$.

Unit fraction

A proper fraction with a numerator of 1, e.g. $\frac{1}{5}$.

Improper fraction

The numerator is equal to or greater than the denominator, e.g. $\frac{7}{5}$.

Mixed number

A whole number and a proper fraction, e.g. $1\frac{2}{5}$.

Iteration

Iteration is a repeated copying of a unit with no gaps or overlaps to form a quantity.

Example: on a number line $\frac{4}{5}$ is located at the endpoint of four units of one fifth.

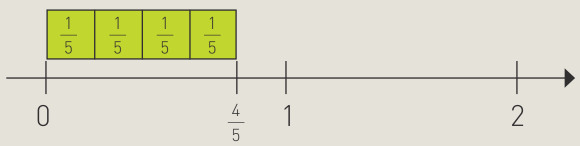


Figure 22

In general, any fraction $\frac{a}{b}$ is made up of units of $\frac{1}{b}$.

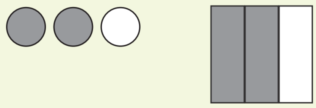
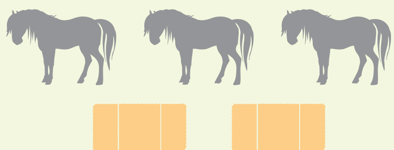
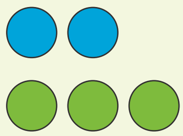
Example: $\frac{4}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$

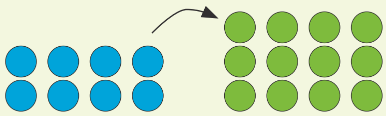
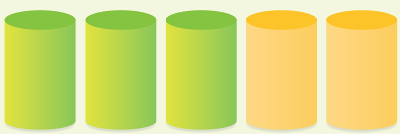
Iteration is important to the understanding of fractions greater than one, such as $\frac{7}{5} = 1\frac{2}{5}$.

Fraction sub-constructs

Fraction sub-constructs are the work of Tom Kieren (Kieren 1980, 1988, 1993) and are detailed in Table 3.

Table 3: Fraction sub-constructs

Sub-construct	Description
Fraction as part–whole	Part–whole is about the relationship between a part and a whole. The whole might be discrete (a set) or continuous (a space). **Example:** two-thirds might be represented as shown below. 
Fraction as quotient	The need for rational numbers comes from division of whole numbers. **Example:** $2 \div 3 = \frac{2}{3}$. If 3 horses share 2 bales of hay equally their share is two-thirds of a bale each. 
Fraction as measure	Measure involves using a quantity as a measure of another quantity. The fraction names the relationship between the unit of measure and the whole being measured. **Example:** the blue Cuisenaire rod measures two-thirds of the green rod.  The blue set measures two-thirds of the green set.  The measure sub-construct is used to create the fraction number line.

Sub-construct	Description
Fraction as operator	A fraction acts as an operator when it works on a quantity through multiplication. **Example:** with two-thirds of 15 ($\frac{2}{3} \times 15 = 10$), two-thirds operates on 15. Operators are also important in finding the relationships between quantities. **Example:** what fraction of 12 is 8? (The problem can be represented as $\square \times 12 = 8$, where $\square = \frac{2}{3}$.) 
Fractions as rates and ratio	Rates and ratios are multiplicative relationships between quantities. Ratios are involved when the same measure is used. A ratio of apple juice to orange juice might be 3:2. The numbers must be measures with the same unit; for example, 3 litres of apple to 2 litres of orange. Note that rational numbers can be used to represent: 1) Part-to-whole relationship **Example:** two-fifths of the ratio is orange juice. 2) Part-to-part relationships. **Example:** the amount of orange juice is two-thirds of the amount of apple juice.  A rate expresses a relationship between different units of measure such as litres and kilometres. **Example:** A car uses 8 litres of petrol when driven 100 kilometres. The car consumes fuel at a rate of 8 litres per 100 kilometres. By dividing each measure by 8 the rate can also be given as 12.5 kilometres per litre.

Strategies for ordering fractions by size

Residual thinking

Residual thinking is using the 'left over' amount when two or more fractions are compared to one, or another benchmark like one-half.

Example: to compare $\frac{4}{5}$ and $\frac{7}{8}$ students might know that four-fifths is one-fifth away from one and seven-eighths is one-eighth away from one.

As one-eighth is smaller than one-fifth, then seven-eighths is closer to one and therefore larger than four-fifths.

Converting to equivalent fractions with the same denominator

Converting to equivalent fractions with the same denominator refers to renaming both fractions so the denominators are the same.

Example: students might compare $\frac{2}{3}$ and $\frac{3}{5}$ by converting both fractions to fifteenths, as shown below.

$\frac{2}{3} = \frac{10}{15}$ and $\frac{3}{5} = \frac{9}{15}$

Since the size of the parts is equal (i.e. both fifteenths), the numerators can be used to show that $\frac{2}{3} > \frac{3}{5}$, since $10 > 9$.

Representations

Physical and diagrammatic representations can be discrete or continuous: discrete representations involve collections of objects; and continuous representations can be partitioned anywhere to create fractions, and include lengths, area, volumes or capacities and mass.

Misconceptions

Most incorrect ideas students possess are the result of overgeneralising the properties of whole numbers and transferring those properties to rational numbers. Below are some examples of common misconceptions.

Ordering by numerators

Given that the numerator tells how many parts, students may incorrectly order fractions by the numerators.

Example:

$\frac{3}{6} > \frac{2}{5}$ since 3 > 2, but $\frac{2}{3} < \frac{4}{8}$ since 2 < 4.

(Note that the first answer is correct by chance since three is greater than two, but the second answer is incorrect.)

Ordering by denominators

Ordering by denominators (or reciprocal thinking) refers to students incorrectly believing that fractions can be ordered by finding the smaller denominator. This misconception arises since the more equal parts a whole is cut into, the smaller the parts become.

Example:

$\frac{3}{6} < \frac{2}{5}$ since 5 < 6, but $\frac{2}{3} > \frac{4}{8}$ since 3 < 8.

(Note that the first answer is wrong, but the second answer is correct by chance.)

Gap thinking

Gap thinking refers to comparing non-unit fractions (i.e. a fraction with a numerator greater than 1) by considering the number of parts rather than the size of the part.

Example: students might incorrectly think that $\frac{5}{6}$ and $\frac{6}{7}$ are the same size, because both fractions require one part to build up to the whole.

They might also think that because $\frac{3}{5}$ is two parts away from $\frac{5}{5}$ then $\frac{3}{5}$ is less than $\frac{1}{2}$ since it is two parts away from a whole, while one-half is only one part away.

Adding numerators and denominators

Given that fractions have whole numbers as numerators and denominators, some students think that addition of fractions works like whole numbers, incorrectly adding numerators together and denominators together.

Example: $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$

Percentages

A *percentage* is a fraction with a denominator of 100. The literal meaning of the % sign is 'per hundred' which comes from the vinculum (line) of a fraction combined with the two zeros from 100.

Applications of percentages

Often percentages refer to a part–whole ratio.

Example: a netball shooter gets 42 goals from 50 shots. Since $\frac{42}{50} = \frac{84}{100}$ her shooting percentage equals 84% meaning 42 out of 50 is equivalent to 84 per 100.

Percentages can be greater than 100, where they represent a comparison of two quantities.

Example: if 40 cats give birth to 70 kittens, the percentage equals 175% because $\frac{70}{40} = \frac{175}{100}$.

Percentages also act as operators.

Example: a 30% discount is offered on all items in a store. An item costing $70 will be discounted by $21 because $\frac{30}{100} \times 70 = 21$.

Strategies

Many strategies are useful for calculating with percentages. Some examples include:

- represent percentages as equivalent part–whole ratios using a dual number line

Example: There are 45 girls in a class of 60 students. The percentage of girls equals 75% because $\frac{45}{60} = \frac{3}{4} = \frac{75}{100}$.

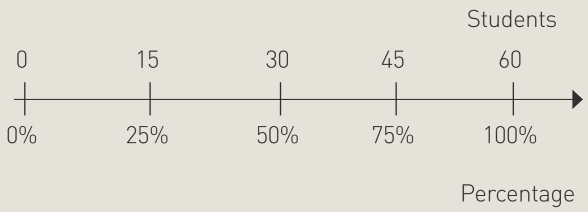


Figure 23

- use 10% as a benchmark

Example: to find 20% of \$140, find 10% of 140 = 14 and double it.

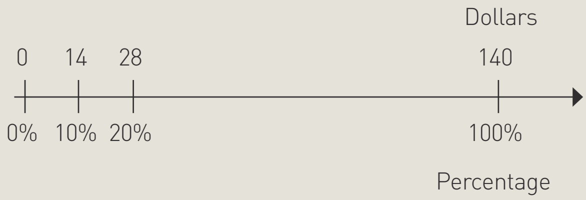


Figure 24

- convert percentages to simple fractions

Example: to find 25% of 48 change 25% to one quarter and calculate $\frac{1}{4} \times 48 = 12$.

- find the unit rate (1%) and multiply (unitary method)

Example: to find 16% of 400 find 1% first and multiply that answer by 16.

1% of 400 = 4 (dividing by 100), so 16% of 400 = 64 (multiplying by 16)

- use common factors to simplify the ratio.

Example: at a tournament there are 48 netballers and 32 AFL players, so there are a total of 80 players; dividing both 48 and 32 by 8 gives 6:4, so the fraction of netballers is 6 out of 10, which is 60%.

Decimals: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in decimals. The specific key ideas in decimals are: **quantity**, **number triad**, **equivalence**, **partitioning**, **base-10 system**, **digit position** and **benchmarks**.

Quantity

A *quantity* is an amount of something which is determined using a number and a unit.

Number triad

A *number triad* represents the relationship between words, symbols and materials or diagrams of a quantity.

Example: 0.86 might be represented as:

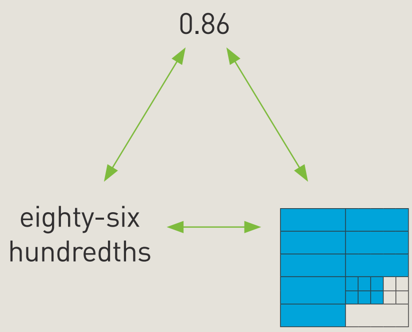


Figure 25

Equivalence

A decimal can be *equivalent* to a fraction or a percentage; for example, a decimal can be expressed as:

- a fraction – 0.5 is the same as one-half
- a percentage – 0.5 is the same as 50%.

Partitioning

A quantity can be separated into parts while maintaining a sense of the whole.

Example: 1.25 = 1 + 0.2 + 0.05 or 1.25 = 1 one + 2 tenths + 5 hundredths

- renamed decimal fractions

Example: 1.25 is the same as 12 tenths and 5 hundredths.

- renaming and regrouping

Example:

0.9 + 0.34 = 1.24 (Rename 0.34)

0.9 + 0.1 + 0.24 = 1.24 (Regroup)

1.0 + 0.24 = 1.24

Figure 26

- compact and expanded decimal forms.

Example: 1.25 = 1 + 0.2 + 0.05 or 1.25 = 1 one + 2 tenths + 5 hundredths

Base-10 system

The *base-10 system* is a number system that is based on grouping quantities in tens or partitioning equally into 10 equal parts.

Each place has a value that is 10 times greater than the place to its right, and one-tenth of the value of the place to its left.

Digit position

The place of a digit in a number determines its value.

There are three ways to interpret a single digit within a decimal fraction. They are *face value*, *place value* and *total value*.

Example: in 0.72 the seven has a face value of 7, a place value in the tenths place and a total value of 7 tenths or 70 hundredths.

Zero can be a placeholder and a digit representing a quantity.

Benchmarks

Benchmarks are trusted quantities or numbers used as reference points to estimate, calculate or compare.

Example: 0.63 is closer to 1 than 0.

Decimals: important concept knowledge

Ragged decimals

According to Roche (2005), *ragged decimal fractions* have a varying number of digits to the right of the decimal point.

Example: 0.4, 0.37, 0.501 and 23.7 are ragged decimals.

(Note that giving students ragged decimals to order supports them to apply place-value knowledge.)

Misconceptions

As occurs with fractions, most misconceptions arise when students overgeneralise the properties of whole numbers and transfer those properties to decimals. Three possible misconceptions that students can have about decimals include: longer is larger, shorter is larger and those who think in terms of money.

Longer is larger

Longer is larger occurs in 'whole-number thinking', such as 4.63 is larger than 4.8 as 63 > 8, and 'column overflow thinking', such as 4.63 is greater than 4.8 as 63 tenths is greater than 8 tenths.

Shorter is larger

Shorter is larger occurs in 'denominator-focused thinking'. A student might incorrectly generalise that one-tenth is bigger than one-hundredth, meaning that any number of tenths is bigger than any number of hundredths. For example, 0.4 is bigger than 0.83.

The shorter-is-larger misconception also occurs in 'reciprocal thinking'. In this case, a student sees the decimal fraction part as the denominator of a fraction, with larger denominators creating smaller fractions. This misconception is revealed when 0.3 is chosen as the larger of 0.3 and 0.4 (as $\frac{1}{3}$ is larger than $\frac{1}{4}$).

In 'negative thinking', a student believes 0.3 is larger than 0.4 as -3 is larger than -4.

Money thinkers

Students who are *money thinkers* have an understanding of the first two decimal places because amounts of money only exist to hundredths of a dollar (cents). They may view decimals as two whole numbers separated by a dot, the first possibly representing dollars and the second cents. It is important to recognise the limitations of teaching decimals through money.

KEY IDEAS *in* MEASUREMENT AND GEOMETRY

Measurement: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in measurement. The specific key ideas in measurement are: **attribute**, **comparison**, **unit of measure**, **conservation** and **equivalence and conversion**.

Attribute

An *attribute* is a characteristic of an object or event.

Example: the measurable attributes of a brick include length, mass, surface area and volume.

Comparison

Similarities and differences of objects can be determined based on particular measurable attributes.

Attributes can be compared based on one or more of the following methods:

- perceptual comparison – attributes of the objects look the same or look different
- direct comparison – objects are placed beside each other to compare the attribute
- indirect comparison – the attribute of objects is compared using a third object

Example: a piece of string or a balance pan can be used to compare objects.

- transitivity – three or more objects are ordered using transitive thinking.

Example: if object A is longer than object B, and object B is longer than object C, then object A must be longer than object C.

Unit of measure

An appropriate unit is used to measure the attribute of an object; a *unit* is a uniform 'piece' of that attribute.

Example: a centimetre is a unit of length. (Note that uniform means that all centimetres are equal.)

Conservation

Conservation is when an attribute of an object changes while the other attributes remain the same. The attribute which has not changed is said to be 'conserved'. Conservation is not applicable to 'time'.

Example: the mass of a piece of clay is conserved whether its form is a sphere or a cylinder.

Equivalence and conversion

The same measure can be expressed using different related units.

Example: there are 100 centimetres in 1 metre; therefore, 2 metres and 35 centimetres is the same as 235 centimetres.

Measurement: important concept knowledge

What is measurement?

The size of a quantity can be measured using counts of a unit.

Example: the mass of an apple may be 250 grams, where 250 is the **count** and **gram** is the unit.

Attributes and units

The following table outlines the attributes and common metric units of measurement.

Table 4: Attributes and units of measurement

Attribute	Common metric units
Length One-dimensional distance of a pathway or object from one point to another	Base unit: metre (m) Sub-units: centimetre (cm), millimetre (mm) Composite units: kilometre (km) Also in common everyday use: decimetre (dm), micrometre (μm) and nanometre (nm)
Area Two-dimensional (flat) space within a bounded region	Base unit: square metre (m^2) Sub-unit: square centimetre (cm^2) Composite units: square kilometre (km^2) Also in common everyday use: hectare (ha), which is 10 000 m^2
Volume Three-dimensional space occupied by an object	Base unit: cubic metre (m^3) Sub-unit: cubic centimetre (cm^3) Composite unit: cubic kilometre (km^3)
Perimeter Distance around the outside of a bounded shape	See units of length *Note: for a circle, the perimeter is called the circumference*
Capacity The amount of gas or liquid able to be enclosed in a space	Base unit: litre (L or l) Sub-unit: millilitre (mL or ml) Composite units: kilolitre (kL), megalitre (ML) *Note: capacity is the volume of the interior of a container*
Mass The amount of matter or substance in an object; not to be confused with the idea of 'weight', which refers to the pull of gravity upon that matter or substance	Base unit: kilogram (kg) Sub-unit: gram (g) Composite unit: tonne (t) (1000 kg) Also in common everyday use: milligram (mg), microgram (μg) and megatonne (Mt)
Time The duration of an event or duration between events The attribute of time can be explored through the following elements: *time telling*: using an instrument to measure time; for example, digital clock, analogue clock, calendar or sand glass *time elapsed*: calculating the duration of time from a starting point to an end point *time span*: daily events (morning, afternoon, evening), tools (calendar, timetable, clock), social or cultural phenomena (Easter, Christmas), and time cycles (millennia, centuries, decades, years, seasons, months, weeks, days)	Base units: second (sec or s), minute (min), hour (h or hr), year (yr) Sub-unit: millisecond (ms) Also in common everyday use: microsecond (μs) nanosecond (ns), day, week, month, decade, century, millennium *Note: many units of time have a base of 12, 24 or 60 and do not conform to the decimal system*
Temperature The heat of a substance	Base unit: degree Celsius (°C)
Angle The amount of turn between two rays that share a common endpoint (vertex)	Base unit: degree (°)

More complex attributes come from relationships between the basic attributes listed in Table 4; for example, speed is the relationship of distance to time and is measured using a rate (kilometres per hour). Other common measurement attributes measured by rates are density, flow and pressure.

Metric system

The metric system forms part of the Standard International (SI) set of standardised measures. Developed by mathematicians and scientists in France during Napoleon Bonaparte's reign, the metric system is based on 10, like the place value system.

The metre was the founding unit in the system and was created to be one ten-millionth of the distance from the equator to the North Pole. It is also the length of a pendulum that completes one swing in one second. Units for other attributes were derived from the metre. One litre is the volume of a cube that has edges of 10 cm. One kilogram is the mass of one litre of water.

In the metric system, prefixes are used to convert the base unit into smaller units or collections of units. The most commonly used prefixes are shown in Table 5.

Table 5: Common prefixes in measurement

Prefix	Meaning	Example
deci-	One-tenth	Decilitre, dL (one-tenth of a litre)
centi-	One-hundredth	Centimetre, cm (one-hundredth of a metre)
milli-	One-thousandth	Millilitre, mL (one-thousandth of a litre)
micro-	One-millionth	Micrometre, μm (one-millionth of a metre)
kilo-	One thousand	Kilogram, kg (one thousand grams)
hecto-	One hundred	Hectometre, hm (one hundred metres)
mega-	One million	Megalitre, ML (one million litres)

Measurement process

Different attributes present variable difficulty for students due to how easily the attributes are sensed and perceived; for example, the attributes of physical space proceed in complexity from length to area, from area to volume, and from volume to capacity. Measurement of mass tends to be easier than measurement of time, since mass is more easily 'felt' than time.

Comparison

The process of comparison is developed in four stages.

1. Objects may be compared visually. Note the difficulty around perception; for example, larger objects appear to be closer.

Example: two pencils are placed beside each other to compare their lengths (note the importance of a common starting point or baseline).

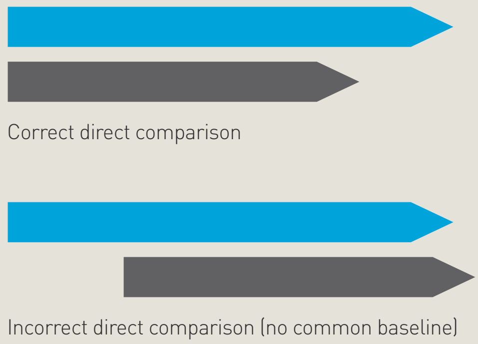


Figure 27

2. Objects may be brought together directly to compare their relative size.
3. Another object may be used to indirectly compare two objects; for example, a length of string may be used to compare the height of a door and a window.
4. A transitive relationship may be established among three or more objects; for example, in Figure 28, if pencil A is shorter than pencil B, and pencil B is shorter than pencil C, then pencil A is shorter than pencil C.

Example:

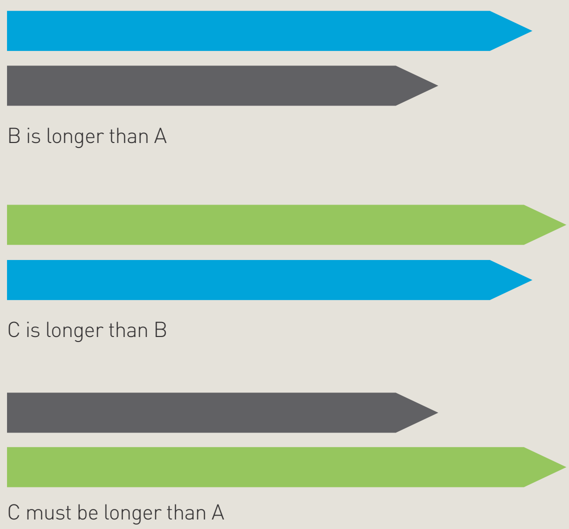


Figure 28

Informal to formal units

Informal units tend to be personal, such as foot lengths, blobs of playdough or handfuls. Formal measures, however, are commonly accepted so that the sizing of units is known and shared by a community; for example, metres, cups and hours.

Students need to be aware that units have the following properties:

1. Units are a piece of the attribute they measure.
2. Units are uniform (i.e. are all of equal size).
3. Units iterate (i.e. a single unit is used repeatedly to find a measurement without gaps or overlaps).

Example:

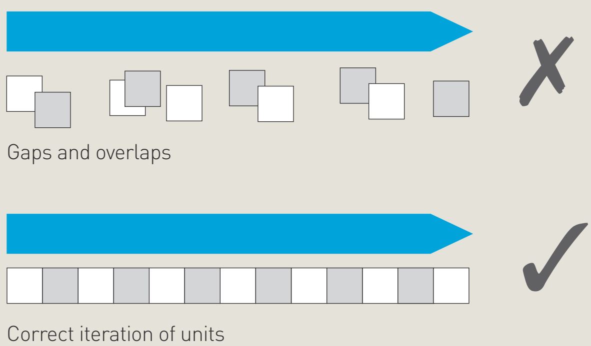


Figure 29

4. Units can be added and subtracted just like other countable objects.

Example: length A, which is 8 cubes long, is combined with length B, which is 6 cubes long, and so have a combined measure of 8 + 6 = 14 cubes.

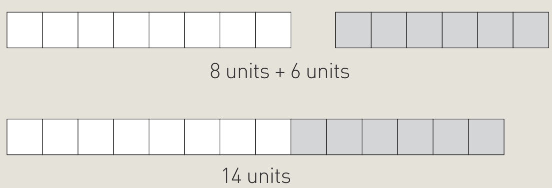


Figure 30

5. Units for the same attribute are related by size in an inversely proportional way. The larger the unit, the smaller the measure. The same pencil might measure 10 white rods or 5 grey rods in length if white rods are half the length of grey rods.

Example:

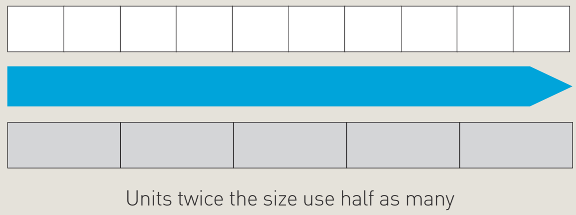


Figure 31

Devices or tools

Measurement scales are created to remove the need to use individual units and to make measurement more efficient. Most devices have scales.

Example: a ruler is a scale and the marks on a measurement jug form a scale.

Students need to be aware that scales have the following properties:

1. Marks show the endpoints of units not the centre of the units.

Example:

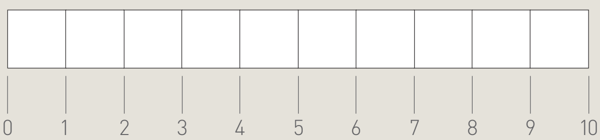


Figure 32

2. Zero marks the start of the scale, although any point on the scale can act as the baseline or arbitrary zero.

Example:

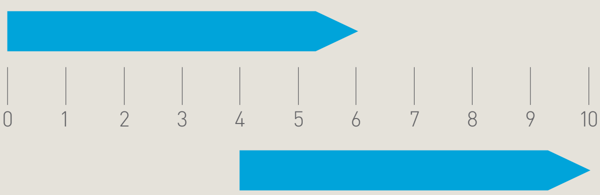


Figure 33

3. Intervals on scales can be equally partitioned into smaller units if more precision is required.

Example: a pencil might be measured in centimetres or millimetres.

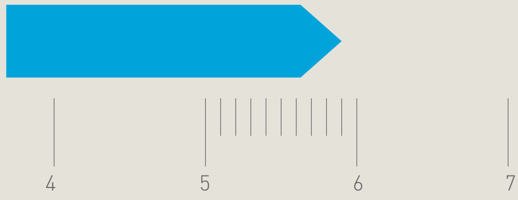
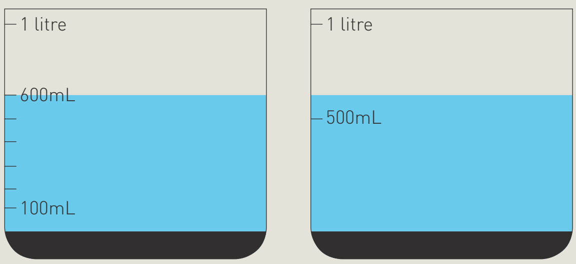


Figure 34

4. Scales are read using a combination of two processes:
   i) iteration: the repeated copying of a trusted measure or interval
   ii) equi-partitioning: the division of an interval into equal more precise intervals.

Example:



Iterating 100mL

Equally partitioning 1L

Figure 35

Calculations with measures, including conversions

Measures can be calculated, calculated with and converted if more appropriate units are required or if attributes are related. The most common types of calculations are:

- finding differences between measures

Example: the temperature at Mount Hotham was -2°C at 5.00 am and rose to 6°C by 2.00 pm. How much did the temperature increase in those 9 hours?

- calculating areas and volumes by multiplication

Example: the area of a fenced enclosure is found by calculating 6 m × 8 m = 48 m^2.

The volume of the object is found by calculating 30 cm × 40 cm × 60 cm = 72 000 cm^3.

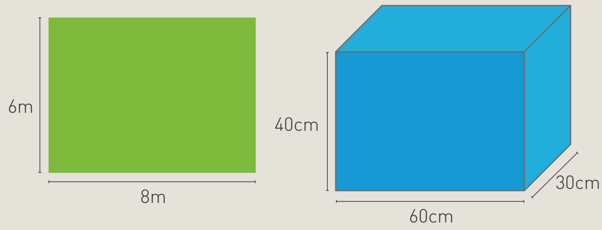


Figure 36

- converting between units of measures for the same attribute

Example: a person is measured as being 165 centimetres in height. Since 100 cm = 1m, their height in metres can be calculated: 165 ÷ 100 = 1.65 m.

- converting between measures for different attributes.

Example A: the interior of an esky measures 30 cm × 40 cm × 60 cm. The volume of the space equals 72 000 cm^3. Since 1000 cm^3 = 1 litre, the capacity of the esky can be calculated: 72 000 ÷ 1000 = 72 L.

Example B: a water cooler bottle has a mass of 0.5 kilograms when empty and holds 15 litres of water. What is its mass when full? (1 litre of water has a mass of 1 kilogram.)

The mass of the full bottle: 15 × 1 + 0.5 = 15.5 kg.

Lehrer's eight key concepts of spatial measurement

Lehrer (2003) has outlined eight key concepts of spatial measurement:

1. **Unit attribute relationship** – units match the attribute being measured
2. **Iteration** – a single unit can be moved to measure a spatial attribute
3. **Tiling** – units fill lines, planes, volumes and angles without spaces
4. **Identical units** – if the units are identical, a count represents the measure, and mixtures of units have to be specified
5. **Standardisation** – formal units are used to facilitate communication
6. **Proportionality** – the size of the unit is inversely proportional to the count of the units, and the larger the unit the smaller the measure
7. **Additivity** – the whole is the sum of the parts
8. **Origin of zero point** – any point can be used as the zero point (for example, the difference between 0 and 10 is the same as between 30 and 40).

Geometry: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in geometry. The specific key ideas in geometry are: **features**, **properties**, **classification**, **orientation**, **transformation**, **point of view** and **symmetry**.

Features

Features, sometimes called attributes, are visual characteristics of a shape or object that can be seen.

Example: a circle and a sphere have 'roundness', a triangle and a pyramid have 'pointiness' (corners or vertices). Squares have straight sides and cubes have flat surfaces (planes). Attendance to features dominates students' early thinking about shapes.

Properties

Shapes and objects have distinguishable characteristics and are named because of their *properties*.

The properties of two-dimensional (2D) shapes typically include relationships among the number, length and relative position of sides, as well as the number and angle of corners, lines of symmetry, convexity and concavity.

The properties of three-dimensional (3D) objects typically include the shape and relative position of faces, surfaces, edges and vertices (corners).

Classes (i.e. categories) of shapes are defined by their properties, which are relationships among features of similar shapes.

Example: equilateral triangles (the class) have three equal sides and three equal angles. All polygons that have those properties belong to the class. (Note that while features are 'seen' in individual shapes, properties are 'thought of,' by connecting common features of collections of shapes and organising those shapes into classes.)

Properties are relationships connecting characteristics of shapes or objects.

Example: a square has four equal sides and four right angles. While the sides and equality are characteristics, connecting them creates a property, as does noticing that the four angles are all right angles.

Some properties are defining; that is, they describe what a shape must have to belong to a class.

Example: a prism is an object made from connected polygons and has a constant cross-section.

Classification

Classification involves establishing criteria to group shapes by their common properties. Classification is about working with relationships among properties, where individual shapes are examples of the classes and are hierarchical.

Example: a quadrilateral is a polygon with four sides. A parallelogram is a quadrilateral that has two pairs of parallel sides. A rectangle is a parallelogram with four right angles. A square is a special type of rectangle that has equal sides. An individual square is an example of a shape that could be classified as a rectangle or a parallelogram or a quadrilateral.

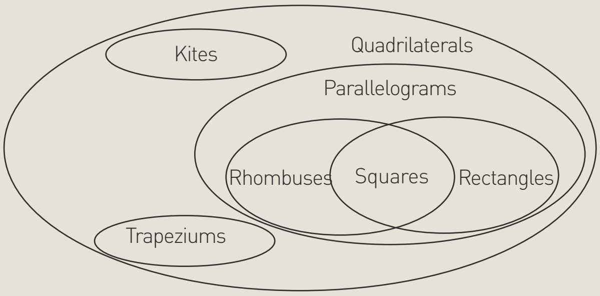


Figure 37 Shape hierarchy

Orientation

Orientation is the position of a shape on a plane or an object in space, in particular the direction that the features of the shape or object are facing.

The orientation of a shape remains unchanged as it is translated (i.e. shifted) or enlarged, but changes when a shape is reflected or rotated through an angle not equal to 360°.

Example: a rotated square is still a square.

Figure 38

Transformation

Transformation is the change in the size, shape or position of a shape or object.

Isometric transformations

Isometric transformations include translations, rotations and reflections. Note: these do not change the size or proportions of a shape or object.

1. Translation (slide): the movement of a shape to a new position. The length of sides and angles remain unchanged.

Example:

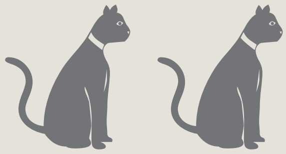


Figure 39

2. Rotation (turn): a change to the position of a shape by rotating it about a fixed point through a given angle. The point may be inside or outside the shape.

Example: 180° rotation about A (external to the shape)

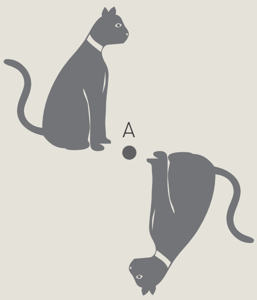


Figure 40

3. Reflection (flip): a change to the position of a shape by reflecting it along a mirror line. The line may run through the shape or be external to it.

Example: a reflection along a mirror line external to the shape

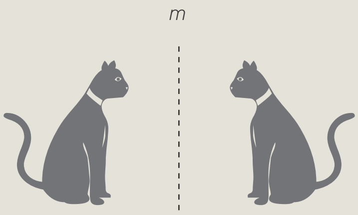


Figure 41

Non-isometric transformations

Non-isometric transformations include enlargements and reductions (dilations).

A dilation is a change to the size of a shape about a point. The lengths of the sides and/or the angles are changed.

Example: an enlargement about P with a scale factor of 2 (lengths of the image shape are double those of the original shape). A scale factor greater than 1 enlarges the shape and a scale factor less than 1 shrinks the shape.

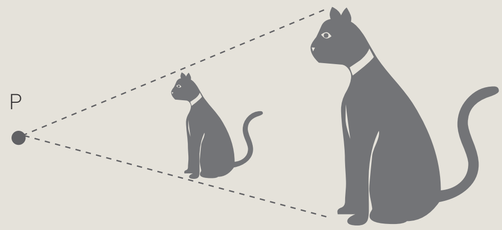


Figure 42

Point of view

Objects appear differently depending upon the position from which they are viewed.

Example: a square-based pyramid will look different when viewed from the top, side or below.

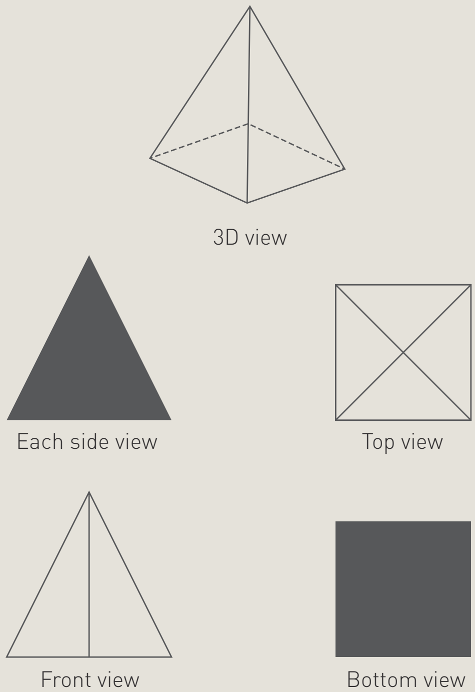


Figure 43

Symmetry

A shape has *symmetry* if it maps onto itself by a transformation, particularly through reflection and/or rotation.

Reflective symmetry

In reflective symmetry, the locations of the mirror are called lines of symmetry.

Example: in the diagram below, *m* is one of two lines of symmetry for the cross section of the letter H.

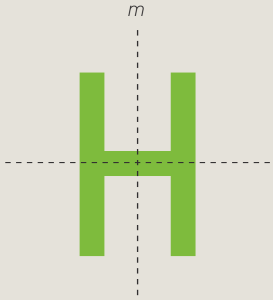


Figure 44

Rotational symmetry

In *rotational symmetry*, the point is known as the centre of rotation and the angle of rotation is the measure of turn that maps the shape onto itself.

Example: the centre of rotation for the recycling logo is P and the angle of rotation measures 120° or one-third of a full turn. The logo has rotational symmetry of order three as it maps onto itself three times in a full turn.

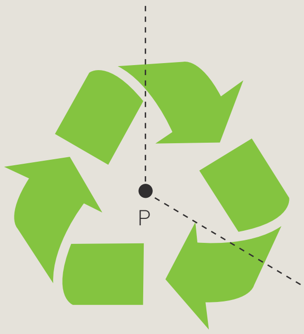


Figure 45

Geometry: important concept knowledge

Geometric reasoning

Geometric reasoning is thinking with the properties of shapes.

Example: students might recognise that if rectangles have four right angles, then squares are a type of rectangle. They might also deduce the value of an unknown angle in a triangle from the two given internal angles.

Point

A *point* is a single location in space. A point is usually represented as a dot, although, theoretically, a point has no area.

Line

A *line* is a set of points that extend infinitely in two directions. A line is usually represented by a straight segment. Theoretically, a line has only one dimension (length) and no area.

Types of lines include:

- horizontal – lines that are parallel to the horizon
- vertical – lines that are at a right angle to a horizontal line
- oblique – lines that are neither vertical nor horizontal
- perpendicular – lines that meet at a right angle.

Ray

A *ray* is a set of points that extend infinitely in one direction. A ray is usually represented by an arrow or vector beginning at a fixed point.

Plane

A *plane* is a flat surface that extends infinitely in two dimensions; width and length.

Two-dimensional shape

A *two-dimensional shape* exists on a flat surface, so it possesses length and width. Two-dimensional shapes include polygons, such as triangles and quadrilaterals, and simple closed curves, such as circles.

Polygon

A *polygon* is a two-dimensional, planar shape that is bounded (i.e. enclosed) by line segments. The line segments form the sides of the polygon. A regular polygon has sides and angles of equal measure.

Three-dimensional object

A *three-dimensional object* exists in real life, so it possesses three dimensions: length, width and depth. Polyhedra, such as prisms and pyramids, and closed surfaces, such as spheres and cylinders, are solid objects that have three dimensions.

Polyhedron

A *polyhedron* is a three-dimensional object bounded (i.e. enclosed) by polygons and called a solid.

Solids or objects enclosed by polygons are called *polyhedra* and those polygons comprise the faces. A regular polyhedron has the same regular polygons for all of its faces.

Example: each of the faces of a cube are squares.

Curved solids, such as cones, spheres, cylinders and others are not classified as polyhedra.

Net

A *net* is a flat shape created by unfolding a three-dimensional solid.

Example: the net below is created by unfolding a cube.

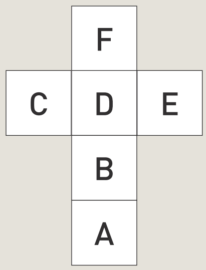


Figure 46

Congruence

Congruent shapes are exactly the same shape and size.

Similar shapes

Similar shapes are exactly the same shape (angles and side ratios), but are a different size.

Regular shapes

A polygon is a *regular shape* if all of its sides are the same length and all of its angles have the same measure.

Tessellation

A *tessellation* is the tiling of a plane in a repeated pattern with no gaps or overlaps of the shapes.

Apex

An *apex* is the highest point above the base of a cone or a pyramid.

Cross-section

A *cross-section* is the flat surface created when a cut is made through an object, parallel to the base. The cross-section of cylinders and prisms are uniform, whereas the cross-section of pyramids and cones are not uniform.

Truncated object

A *truncated 3D object* has a vertex removed by cutting along a plane.

Right object

A 3D object is 'right' when the top face or the apex is centered above the base and is perpendicular to the centre point of the base.

Oblique object

An *oblique object* is a 3D object that does not fulfill the criteria for 'right'.

Angle

An *angle* is a figure formed by two rays joining at a common endpoint (P), which is used to represent a turn of one ray from another about P. Angles assist in defining the properties of classes of shapes.

Example: a rectangle is a quadrilateral with equal angles.

Adjacent angles

Adjacent angles are two angles next to each other that share a common ray.

Example: angles *a* and *b* are adjacent, as are angles *b* and *c*, angles *a* and *d* and angles *c* and *d*. Each adjacent pair adds up to 180°.

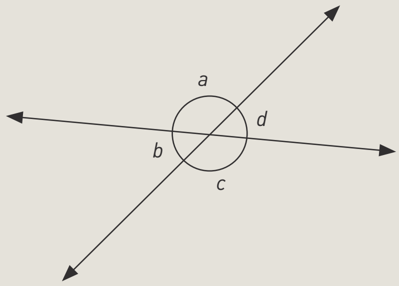


Figure 47

Complementary angles

Complementary angles are adjacent and add up to 90°.

Example: 40° is the complement of 50° and vice versa.

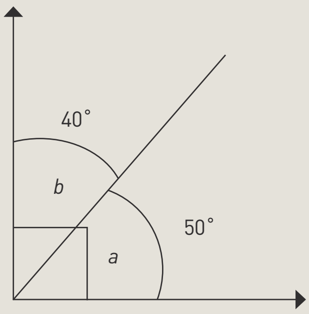


Figure 48

Supplementary angles

Supplementary angles are two adjacent angles that add up to 180°.

Example: 55° is the supplement of 125° and vice versa.

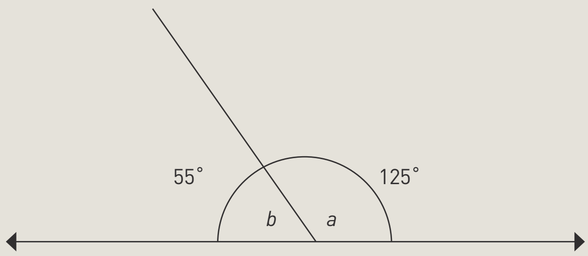


Figure 49

Vertically-opposite angles

Vertically-opposite angles refer to two lines intersecting so that the angles opposite to each other are the same measure.

Example: angles *a* and *c* are opposite, as are angles *b* and *d*, so *a* = *c* and *b* = *d*.

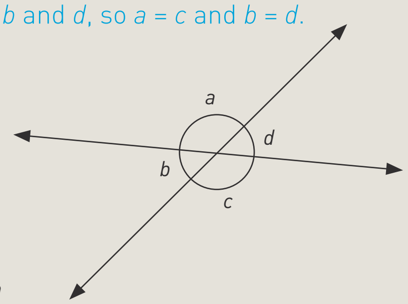


Figure 50

Angles at a point

Angles at a point refers to angles surrounding a point (with no gap or overlap) that add up to 360°.

This property is particularly important for establishing which polygons, or sets of polygons, will tessellate.

Example: in the tessellation below, the angles meeting at each vertex (i.e. point) add to 360°.

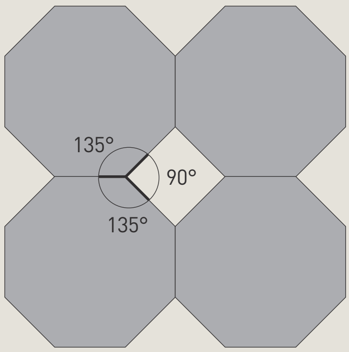


Figure 51

Parallel lines

Parallel lines extend infinitely in two directions but never meet. A real-life analogy is railway tracks.

Example: AB and XY are representations of parallel lines.

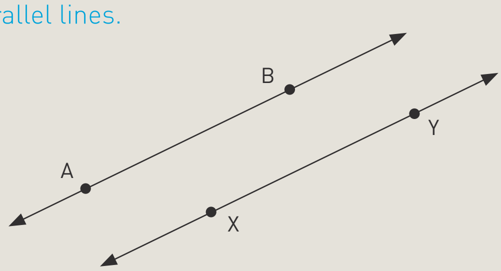


Figure 52

Transversals

A *transversal* is a line that intersects parallel lines, forming different sets of angles.

Example:

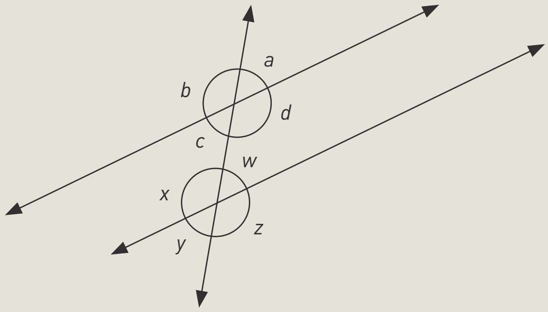


Figure 53

Corresponding angle

A *corresponding angle* is an angle that is on the same side of the transversal and in a like position. Corresponding angles are equal in measure.

Example: in Figure 53, angles *a* and *w* are corresponding, as are *d* and *z*, *b* and *x*, *c* and *y*, so *a* = *w*, *d* = *z*, *b* = *x* and *c* = *y*.

Alternate angle

An *alternate angle* is an angle that is on the opposite side of the transversal and inside the two intersected lines. Alternate angles are equal in measure.

Example: in Figure 53, angles *c* and *w* are alternate, as are *d* and *x*, so *c* = *w* and *d* = *x*.

Co-interior angle

A *co-interior angle* is an angle that is on the same side of the transversal and inside the two intersected lines. Co-interior angles add up to 180°.

Example: in Figure 53, angles *d* and *w* are co-interior, as are *c* and *x*, so *d* + *w* = 180° and *c* + *x* = 180°.

Location: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in location. The specific key ideas in location are: **position**, **direction** and **orientation**.

Position

The *position* of an object on a plane or in space can be specified and described relative to a reference point.

Direction

The *direction* (i.e. movement) of an object can be described relative to its starting position and surrounding landmarks.

Orientation

The *orientation* of an object can be described in relation to a reference direction.

Usually the reference direction is a compass point or bearing, or axes on a Cartesian plane.

Location: important concept knowledge

Visual representations

The location of objects on a plane and/or in space can be visually represented; each representation has particular purposes and ways in which they are read and interpreted.

Visual representations include diagrams, plans, grids, maps, directories and networks.

Coordinate system

A *coordinate system* is a system used to locate the position or direction of an object on a plane or in space relative to the distance from an origin.

Grid reference

An *alpha-numeric* grid reference defines a region on a map.

Example: in the alpha-numeric reference D2, the letter D represents the horizontal *x*-axis and the numeral 2, represents the vertical *y*-axis.

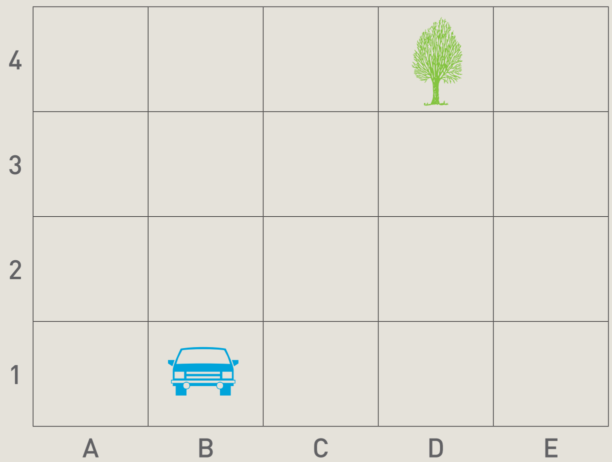


Figure 54

Grid maps

Overlaying an array of squares onto a map provides a means to identify the location of landmarks. The squares are individual (discrete) so the landmark lies inside the given space; for example, the car is located at B1 and the tree is located at D4. The horizontal reference is given first.

An *ordered pair* reference defines a point, rather than a region, on a map or Cartesian plane.

Example: in the ordered pair reference (2,3), the first number, 2, represents the *x*-coordinate and the second number, 3, represents the *y*-coordinate on the Cartesian plane; the Cartesian plane is divided into four quadrants.

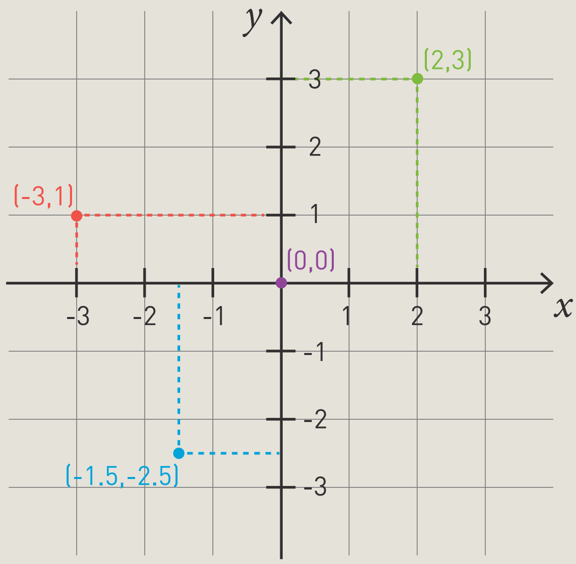


Figure 55 Example of a Cartesian plane

Compass points

Cardinal direction

A *cardinal direction* is a bearing described by north, south, east and west, and is commonly denoted by the direction's initial.

The four main points on a compass are:

- north (N) – 0° or 360°
- east (E) – 90°
- south (S) – 180°
- west (W) – 270°.

Inter-cardinal direction

Inter-cardinal points can also be used:

- north-east (NE) – 45°
- south-east (SE) – 135°
- south-west (SW) – 225°
- north-west (NW) – 315°.

Other inter-cardinal points include:

- north-north-east (NNE) – 22.5°
- east-north-east (ENE) – 67.5°
- east-south-east (ESE) – 112.5°
- south-south-east (SSE) – 157.5°
- south-south-west (SSW) – 202.5°
- west-south-west (WSW) – 247.5°
- west-north-west (WNW) – 292.5°
- north-north-west (NNW) – 337.5°.

Bearings

A *bearing* is the angle between north and another landmark as taken from a fixed point. The angle is measured in a clockwise direction.

Example: to reach the tree the person needs to walk at a bearing of 120°.

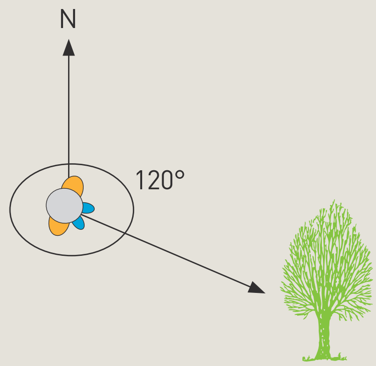


Figure 56

BOLTSS

BOLTSS (i.e. border, orientations, legend, title, scale and source) is standard information to include on a map:

- border – the limits of the area covered by a map
- orientation – an indication of north
- legend – symbols on the map that represent natural or artificial features on the ground; legends are sometimes referred to as 'keys' as they 'unlock' the meaning of the symbols
- title – the name of the map, identifying the geographic area that it covers

Example: 1 cm: 500 m. Each centimetre on the map is equivalent to 500 metres on the ground.

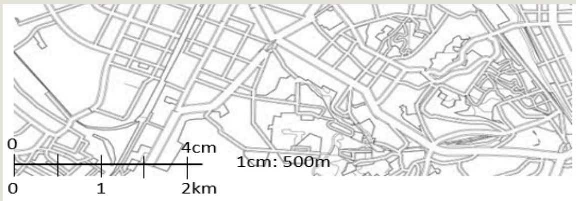


Figure 57

- scale – a ratio indicating distance on a map in relation to distance on the ground
- source – where the information on the map originated.

Example:

Docklands

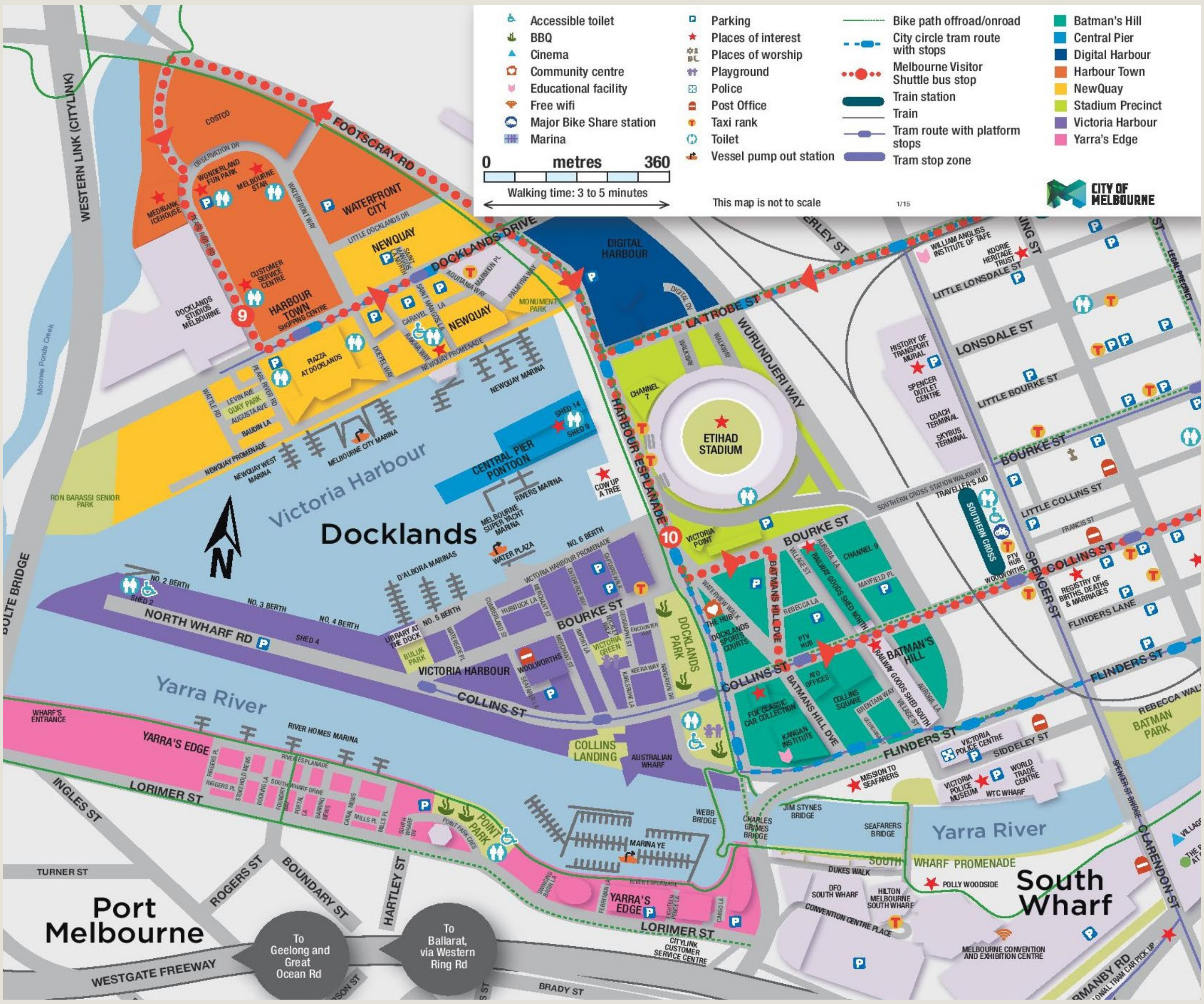


Figure 58 (Source: onthemap.com)

KEY IDEAS *in* STATISTICS AND PROBABILITY

Chance: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in chance. The specific key ideas in chance are: **probability**, **randomness**, **fairness**, **bias**, **independent event** and **dependent event**.

Probability

Probability is a measure of likelihood. It quantifies the chance of an event occurring.

Example: when flipping a coin there is a 50% chance of it landing on heads.

Experimental probability is calculated by the frequency of an event occurring based on repeated trials.

Example: flipping a coin 30 times

Theoretical probability is determined by systematically finding all the possible outcomes of an experiment (the sample space).

The number of favourable outcomes is compared to all the possible outcomes to express the probability as a fraction, decimal, percentage or ratio. The more trials within an experiment the more experimental probability aligns to theoretical probability.

Randomness

Randomness is the unpredictability of an outcome occurring. It is not possible to predict which outcome in a trial will occur because randomness is not influenced by any factor other than chance.

Example: a roll of a dice has six possible outcomes. It is not possible to predict which outcome will occur.

Fairness

Outcomes are fair when there is an equal chance of occurrence.

Example: the toss of a coin is fair because heads or tails are equally likely.

A weighted dice is not 'fair' because the possible outcomes do not have an equal chance of occurring.

Bias

Biased outcomes do not have an equal chance of occurrence. They are not fair.

Example: the spinner below has a greater likelihood of landing on 2. It is random but not fair.

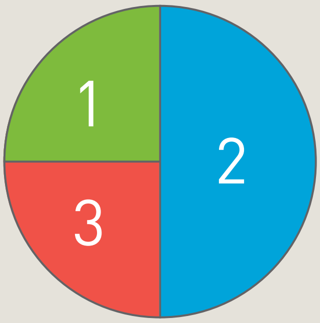


Figure 59

Independent event

An *independent event* is an event that is not affected by the outcome of another event.

Example: when a coin is tossed, it is equally likely to land on either heads or tails in subsequent tosses.

Dependent event

A *dependent event* is an event that is affected by the outcome of a prior event.

Example: the probability of drawing a red counter from a bag holding 5 red counters and 5 black counters is affected by whether the counters are replaced or not replaced after previous draws (Van de Walle et al. 2010).

Chance: important concept knowledge

Describing chance

The probability of events can be described using language and/or numerical terms.

Language descriptions

Words can be used to describe and order probabilities.

Example: 'Impossible' describes an event that can never occur, whereas 'certain' describes an event that will always occur.

Impossible — Unlikely — Equally likely — Likely — Certain

Figure 60

Numerical descriptions

Numbers from zero to one can be used to express probabilities. A probability of zero describes an event that is impossible to occur. A probability of one describes an event that is certain to occur. A probability of one-half describes an event that has the same chance of occurring as the chance of it not occurring.

Example:

0 .... $\frac{1}{2}$ 1

0% .... 50% 100%

Figure 61

Ratios are sometimes used to represent the odds of an event occurring.

Example: the odds ratio of getting a head with a single coin toss is 1:1, since there is one outcome that gives a head to one outcome that does not. The odds of getting 6 with one dice roll is 1:5.

Experiment

An *experiment* is an enactment of a situation.

Example: carrying out 30 trials of rolling two dice and finding the totals each time

Trial

A *trial* is a particular performance of an experiment.

Example: a single roll of two dice

Outcome

An *outcome* is one possible result for a single trial.

Example: the red dice may land on 6 and the blue dice may land on 4.

Frequency

Frequency refers to the number of times an outcome occurs.

Sample space

The *sample space* is the set of all the possible outcomes of an experiment.

Event

An *event* is a subset of the sample space for a random experiment (ACARA 2019).

Example: what are the chances of getting a total of 5 when two dice are rolled? Getting a total of 5 is an event. There are four outcomes that result in that event: (1,4), (2,3), (3,2) and (4,1).

Representations

There are several ways to systematically determine the number of possible outcomes (i.e. sample space) for situations involving elements of chance.

Consider the situation of a game played with a paper cup containing four marbles (two white and two black). Two marbles are drawn out. The following are representations of the possible outcomes.

Systematic list

In a systematic list, the most important step is to label each marble individually then match the marbles into pairs systematically. That way, no possible pairings are missed. Note that the order of marbles coming out is considered:

- W1W2, W1B1, W1B2
- W2W1, W2B1, W2B2
- B1W1, B1W2, B1B2
- B2W1, B2W2, B2B1.

Tree diagram

In a *tree diagram*, the end of each arm is a different outcome (see Figure 62). Note that each marble has an individual code.

Example:

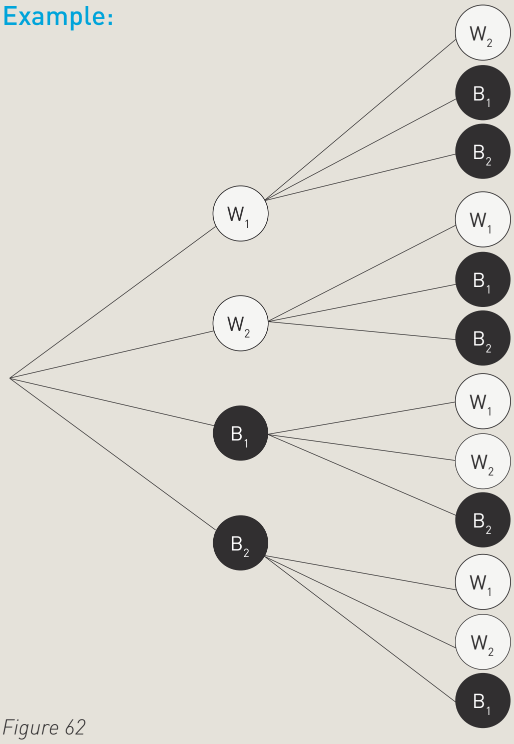


Figure 62

Table

In a *table* (sometimes called a matrix), each cell represents a different outcome.

Example:

	W_1	W_2	B_1	B_2
W_1		W_2W_1	B_1W_1	B_2W_1
W_2	W_1W_2		B_1W_2	B_2W_2
B_1	W_1B_1	W_2B_1		B_2B_1
B_2	W_1B_2	W_2B_2	B_1B_2	

Figure 63

Misconceptions

Subjective judgements

Students often describe the chance of an event in terms of 'personal feelings'.

Example: in a board game, students might believe that they cannot roll a 6 because 'they are very unlucky', or 'never win anything'.

Recency effect

Students can mistakenly make predictions about the likelihood of an independent event based on the outcome of the previous trials.

Example: a person who has just bought a new white car, notices more white cars on the road and expects to see white cars more frequently.

Independence effect

Students might mistakenly make predictions about the likelihood of an independent event based on the outcome of the previous trials. Chance has no memory.

Example: when tossing a coin, on the basis of previous independent events, the student might think that because there have been three heads in a row, the next toss should be tails.

Sampling variability effect

Variations in the results of an experiment due to sampling often cause students to question their theoretical model.

Example: a student might predict that the spinner below will land on yellow half of the time. However, after 12 trials, the spinner lands on yellow only four times. The student questions their belief that there is a half chance of getting yellow on each spin.

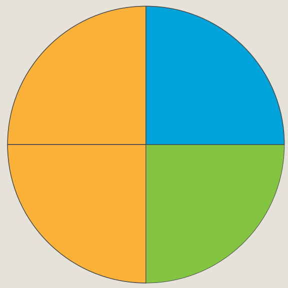


Figure 64

Equi-probability bias

Students can mistakenly believe that every chance situation is fair and has equal probabilities. However, in more complex situations, events often have unequal probabilities.

Example: when rolling a single dice, each number from 1 to 6 is equally likely, however, when rolling two dice and finding the total, students often incorrectly assume that each total of 2 to 12 is equally likely.

Outcomes equal events confusion

Related to the equi-probability bias, students often confuse outcomes with events. In the two-dice scenario, the event of getting a total of 7 has six associated outcomes. If the dice are assigned labels first and second, the outcomes are (1,6), (2,5), (3,4), (4,3), (5,2), (6,1). The outcome (3,4) is not the same as the outcome (4,3).

Data investigation and interpretation: key ideas

The overarching key ideas of estimation, benchmarks, visualisation, equality and equivalence, language and strategies need to be considered when developing units of work in data investigation and interpretation. The specific key ideas in data investigation and interpretation are: **classification**, **variation**, **expectation**, **distribution**, **randomness** and **informal inference**.

Classification

Classification involves making decisions about the categorisation of data. Data is often sorted into categories by characteristic.

Example: pets might be sorted into dogs, cats, birds and so on; and test scores might be sorted into class intervals, such as 0–10%, 11–20%, 21–30% and so on.

The following five key ideas are interrelated.

Variation

Variation describes the differences observed around us in every measurable aspect of life, such as age, height, eye colour and temperature. Variation is fundamental and directly connected to the other four key ideas.

Example:

the daily maximum temperature varies over a year

heights of students vary (i.e. natural deviation)

students might vary in the accuracy with which they measure their height (i.e. measurement variation)

a sample of students might vary in terms of 'middle height' (i.e. sampling variation)

Expectation

Expectation is a prediction based on patterns and differences in data.

Example: 'I expect that the highest daily temperature will occur in February', or 'I expect that most Year 6 students are usually taller than most Year 3 students'.

Distribution

Distribution is a shape or relationship that represents a whole dataset. Common features of the shape of a distribution include centre, symmetry and non-symmetry (i.e. skewness), most frequent values or categories, and spread. Categorical data might be represented by a bar graph that shows how the data is distributed across the categories.

Example: the highest frequency in the graph below is in the meerkat category and the lowest is in the chimpanzee category.

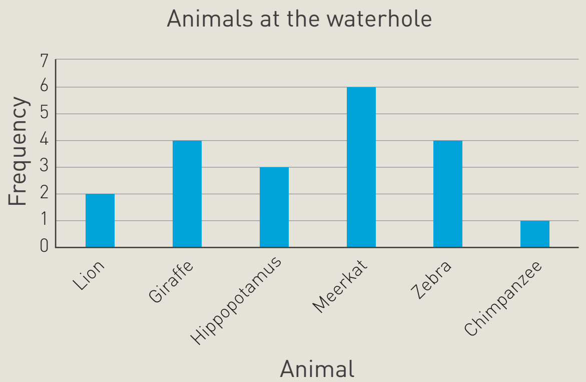


Figure 65

Numerical data might be displayed in a dot plot.

Example: the distribution below is mostly grouped around 55–75 beats per minute.

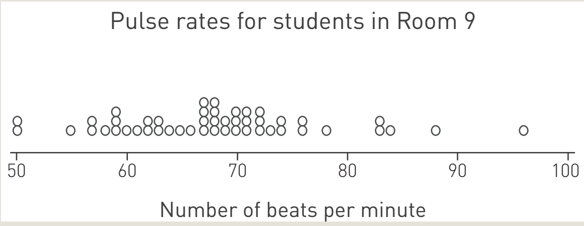


Figure 66

Randomness

Randomness occurs when all possible outcomes of a situation have an equal chance of being selected.

Example: a random process, such as rolling a dice, requires that all six numbers have an equal chance of being drawn.

Informal inference

An *informal inference* is a generalised claim that is formulated from the data collected (Watson n.d.).

Example: generally, February has the highest temperature.

Data investigation and interpretation: important concept knowledge

Statistical inquiry

Statistics is the process of answering questions using data. The data may need to be collected or may already exist. Five interconnected stages make up the statistical inquiry cycle, as shown in Figure 67.

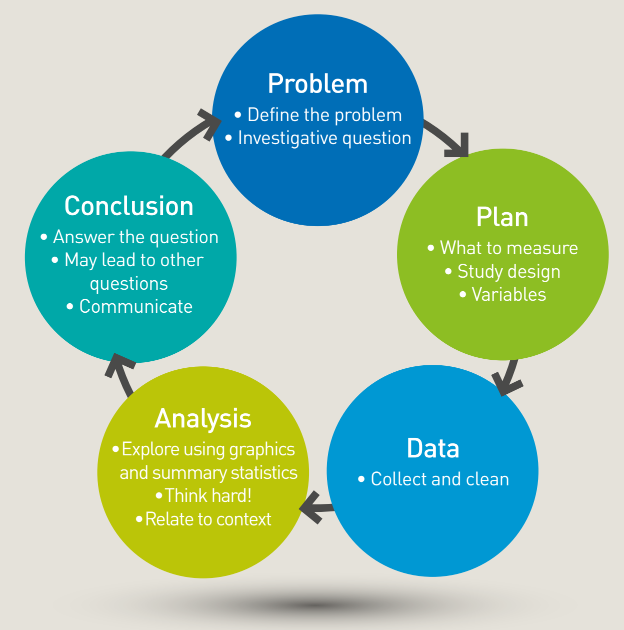


Figure 67 (Adapted from https://ecstep.com/statistics)

Problem

Inquiry begins with an issue or *defining the problem*. The inquiry problem is refined into inquiry questions. Types of questions include:

- summary (for example, how tall are seven-year-old students?)
- comparison (for example, are seven-year-old girls taller than seven-year-old boys?)
- relationship (for example, is there a relationship between children's heights and their arm spans?).

Plan

Planning involves considering what data is needed to answer the inquiry question and how it is to be collected. Sometimes the data already exists and can be accessed without the need to collect new data. The critical consideration is whether the data can be used to answer the inquiry question.

Data

Data involves the collection of data and the representation of the collected data into a communicable format. Data can be organised into two types: catagorical and numerical.

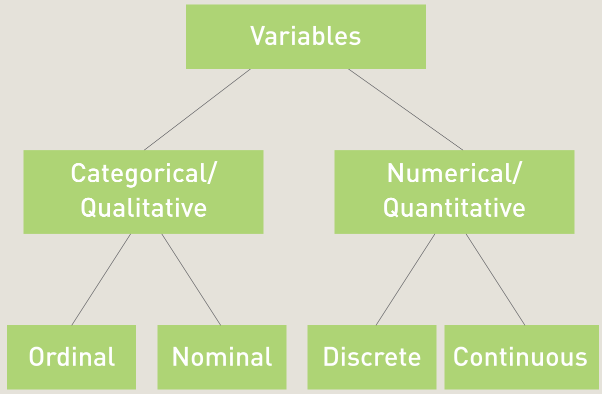


Figure 68

Categorical data, also known as qualitative data, may be represented by a name, symbol or a number code. Below are some examples of data types.

- *Nominal data* is a set of data that can be separated into distinct grouping or categories that cannot be organised in a logical sequence.

Example: different colours of jelly beans in a packet

- *Ordinal data* is a set of data that can be logically ordered or ranked.

Examples: clothing size (S, M, L), academic grades (A, B, C) or scale scores (1–5)

Numerical data, also known as quantitative data, is data that can be expressed as counts (numbers) and specific measures (units).

- *Discrete data* is a set of data that can take distinct and specific number values.

Example: the number of goals scored by a football team or shoe sizes

- *Continuous data* is a set of data consisting of measurements that can take on any decimal value along a continuous scale.

> **Example:** the height of Year 3 students (Note that continuous data is expressed to an elected precision, such as time to the nearest minute, price to the nearest dollar or height to the nearest centimetre.)

Data collection tools

Data can be collected using a range of tools. These may include:

- surveys
- questionnaires
- interviews
- observations
- measurements
- experiments (usually associated with probability).

Data displays

Data displays are a tool for investigation, as well as a way to communicate findings. The choice of data display should be directly related to the investigation and the type of data collected.

All data

Tally chart

Each mark in the tally chart below represents one participant's hair colour. A count of five is shown as four vertical lines crossed with a diagonal line.

Example:

Hair colour	Tally
Black	𝍸 II
Brown	𝍸 𝍸
Red	𝍸
Blonde	𝍸 II
Pink	I

Figure 69

Frequency table

Frequency refers to the number in a category. In the frequency table in Figure 70, 10 people surveyed had brown hair.

Example:

Hair colour	Tally	Frequency
Black	𝍸 II	7
Brown	𝍸 𝍸	10
Red	𝍸	5
Blonde	𝍸 II	7
Pink	I	1

Figure 70

Categorical data

Pictographs

Each symbol in a *pictograph* refers to a number within a category.

Example: three participants in the sample wore a yellow hat.

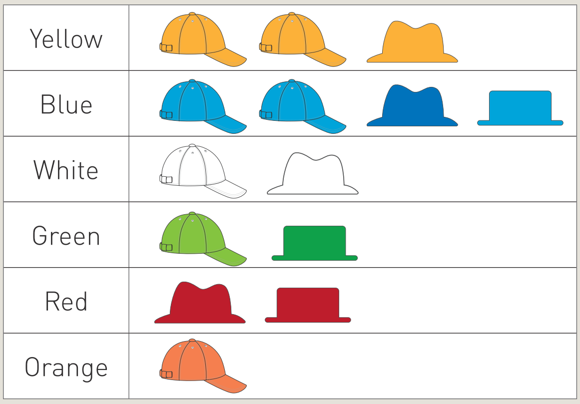


Figure 71

Pictographs with a key usually have icons that represent more than one item within a category. In the pictograph in Figure 72, each drop represents the blood type of four people.

Example:

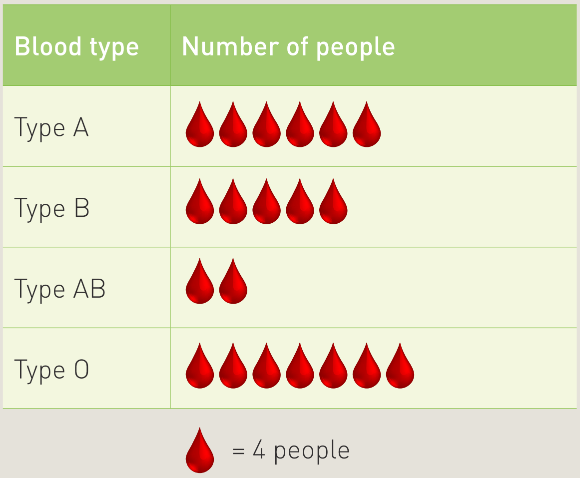


Figure 72

Column graph

A *column graph* (also known as a bar graph) represents the frequency within each category. Column graphs can be oriented either vertically or horizontally. In the column graph shown in Figure 73, 40 people sampled had brown eyes.

Example:

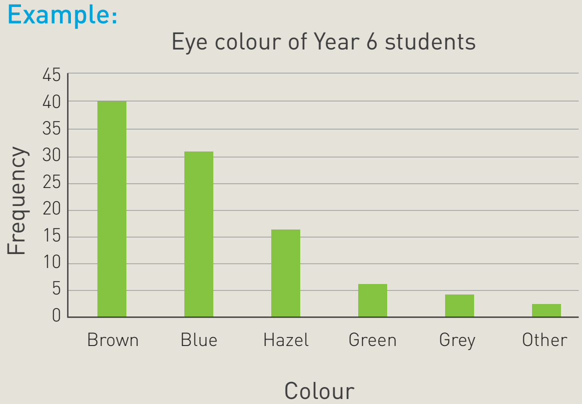


Figure 73

Pie graph

In a *pie graph*, the size of each sector shows the proportion of people/items in that category compared to the entire group.

Example:

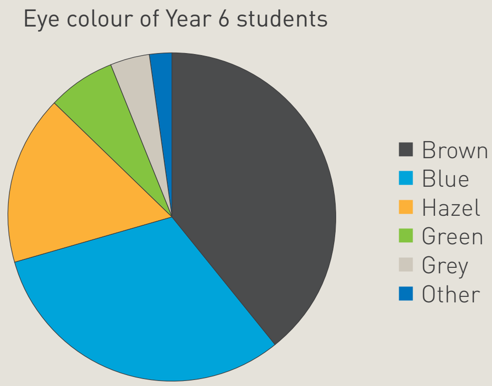


Figure 74

Numerical data

Stem-and-leaf plot

The *stem-and-leaf plot* in Figure 75 shows 13 students' estimates for the mass of an apple in grams. (Note that the stems are the place values in tens and the leaves are the ones digits of the estimates.)

Example: 15|5 represents 155

Stem	Leaf
10	0
11	
12	0
13	
14	0
15	5
16	0 0 5
17	0 5 6
18	0
19	0
20	
21	0

Figure 75

Dot plot

Each dot on a *dot plot* represents the data of one item, so the height of each line of dots represents the number of items with that data value.

Example: two mothers have an age of 45 years.

Dot plots are used to display discrete numerical data, like what's shown in Figure 76.

Example:

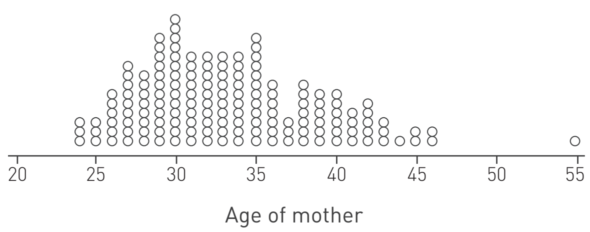


Figure 76

Box and whisker plot

Note: Box and whisker plots are introduced in the Year 9 curriculum. The following has been included as supplementary information.

The box and whisker plot for the data of mothers' ages in Figure 77 gives five measures of the distribution of ages. The extreme ends of the whiskers are the lowest and highest ages. The central measure is the median, and the left and right ends of the box are the lower and upper quartile (LQ and UQ). This means that 50 per cent of the data lies inside the box.

Example:

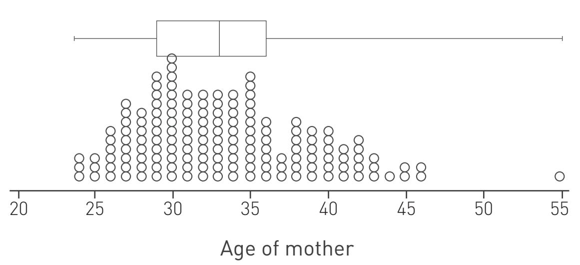


Figure 77

Box and whisker plots can be used for both discrete and continuous numerical data.

Quartiles are the result of splitting a distribution into quarters. The lower quartile (LQ) is the value at which one-quarter of the data values are below. The upper quartile (UQ) is the value at which one-quarter of the data values are above. The interquartile range is the difference between upper and lower quartiles.

Histogram

A *histogram* represents discrete or continuous numerical data. The height of each column represents the frequency with the matching range of values.

Note that the graph in Figure 78 is not the same as a column graph. The columns are joined as the numerical data is continuous.

Example:

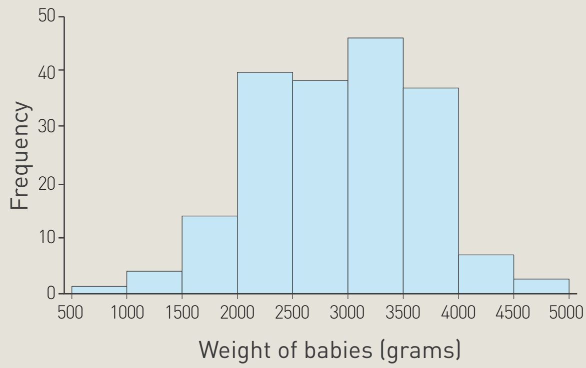


Figure 78

Scatterplot

A *scatterplot* represents the relationship between two variables. Each point represents a pair of values for a single person/item; in Figure 79 foot length and height are shown, both measured in centimetres. A line of best fit (i.e. a regression line) through the points is used to establish a correlation between the variables. The closer the points to the line, the stronger the correlation between variables.

In Figure 79, the line would have a positive slope and show that people with longer feet tend to be taller.

Example:

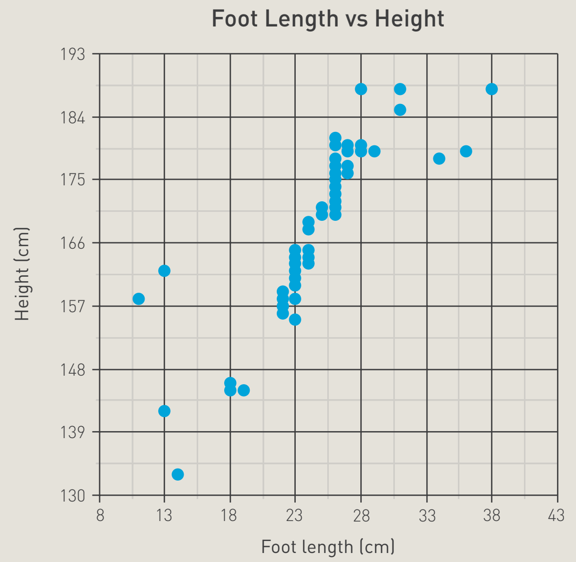


Figure 79

Line graph

A *line graph* is used to display time-series data. The example in Figure 80 shows how the times for the 100-metre men's world record have changed from 1900 to 2020. Line graphs are used to look for trends and patterns over time.

Example:

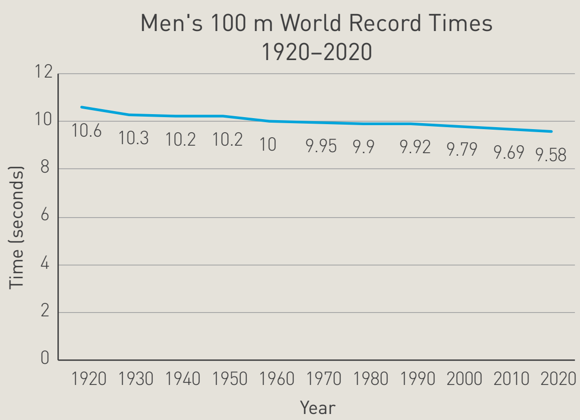


Figure 80

Comparison of groups

Statistical investigations often involve looking for similarities and differences among groups. The data displays shown below can be used for comparison. On the top line, 6|11|8 represents the resting pulse rate of 116 for netball players and the resting pulse rate of 118 for AFL players.

Example: a back-to-back stem-and-leaf plot might be used to compare the fitness of netball and AFL players, as measured by resting pulse rate.

Netball Players		AFL Players
6	11	8
	10	0 4
	9	2 2 6 6
6 4 4 0 0 0 0 0	8	0 0 0 4 8 8 8
6 6 6 2 2 2 2 2	7	2 2 6 6 6 6
8 8 8 4 0 0	6	0 0 4 4 8 8
	5	2 2 2 6 6

Figure 81

Column graphs can be used to compare different groups. The side-by-side graph in the example in Figure 82 compares the favourite fast food of two groups: boys and girls in Year 5.

Example:

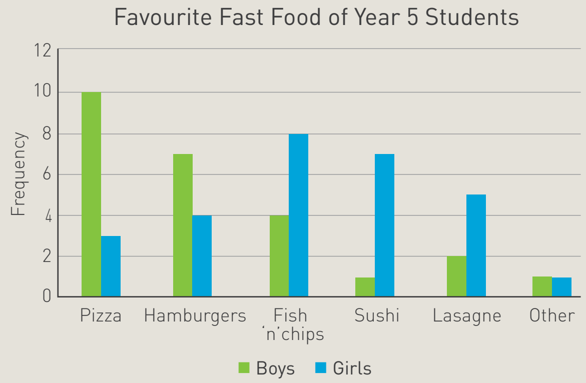


Figure 82

Appearance of data

- Symmetrical: the mean, median and mode are close together.
- Skewed: the mean, median and mode are not close together.

When comparing groups, distributions usually overlap, so measures of central tendency, such as the median or mean, are used to represent each group.

In the graph below, the scores of female and male Australian students on a memory test are compared. The median score for males is 43 and for females is 40, showing an insignificant difference.

Example:

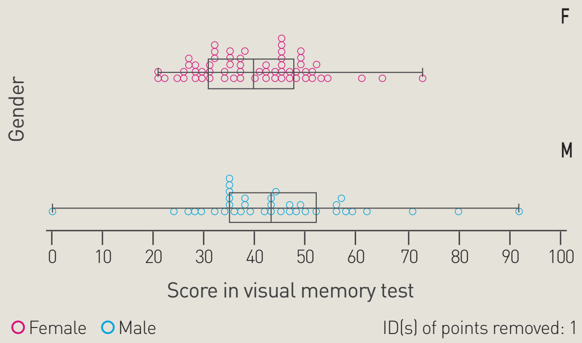


Figure 83

Analysis

Data analysis is the process of making sense of the data with respect to the inquiry question. Statisticians use displays, such as graphs and tables, and measures, such as medians and ranges, to look for patterns (consistencies), differences among groups, and trends (patterns over time).

Data measures

Data measures are calculated to represent a single feature of a whole dataset. Usually the dataset is composed of numbers. Mean and median are measures that represent the centre of a dataset, whereas range and interquartile range (IQR) measure spread.

Measure of variation

The *measure of variation* refers to the spread of data and includes the:

- range – the difference between the greatest and smallest values in the dataset
- interquartile range – the difference between the upper and lower quartile in the dataset. It represents the middle 50% of the dataset.

Measures of central tendency

Measures of central tendency refer to averages, such as the:

- mean – the sum of all values in the dataset divided by the number of data points (e.g. scores or values)
- median – the middle value in an ordered set of numeric data
- mode – the value that occurs most frequently in the numeric dataset.

The shape of distributions is also used to compare groups and to describe single distributions. A normal distribution is bell shaped. However, distributions can be skewed positively or negatively, can have more than one mode and can even be rectangular.

Conclusion

The conclusion is an answer to the inquiry question that is supported by the data. Context informs the significance attached to findings.

The data is interpreted to develop inferences in relation to the original investigation and the findings are communicated.

Findings may lead to other questions that require further investigation, prompting a new data inquiry cycle.

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